H. Furstenberg applied the phenomenon of multiple recurrence for measure-preserving transformations [J. Anal. Math. 31, 204–256 (1977; Zbl 0347.28016)] to reprove Szemerédi’s theorem on arbitrarily long arithmetic progressions in sets of integers of positive upper density. This was done using the technique nowadays known as Furstenberg’s correspondence, which is naturally formulated in terms of amenable groups. The correspondence consists in the following. Given a sequence of Følner sets \((F_n)\) in a (countable, for simplicity) amenable group \(G\), and a subset \(E \subseteq G\) whose upper density with respect to the sequence \((F_n)\), \(\bar{d}(E) = \limsup |E \cap F_n|/|F_n|\) is strictly positive, there exist a Lebesgue probability space \((X, \mu)\) equipped with a measure-preserving action of \(G\) and a measurable subset \(A \subseteq X\) such that \(\mu(A) = \bar{d}(E)\). Moreover, for some subsequence \((n_k)\) of natural numbers and for every collection \(g_1, g_2, \ldots, g_r \in G\) one has

\[
\mu(A \cap g_1^{-1}A \cap \cdots \cap g_r^{-1}A) = \lim_{k \to \infty} |E \cap g_1^{-1}E \cap \cdots \cap g_r^{-1}E \cap F_{n_k}|/|F_{n_k}|.
\]

In addition to the original construction by Furstenberg, based on symbolic dynamics, there are today various other known realizations of the Furstenberg correspondence (reviewed in the paper). For example, one can extend any left-invariant finitely additive measure generated by the given sequence of Følner sets to a regular countably-additive Borel measure on the Stone-Čech compactification of the group, take as \(A\) the closure of \(E\) in \(\beta G\), and then pass to a suitable standard Lebesgue quotient of the non-separable probability space \(\beta G\) in a \(G\)-equivariant way.

The main result of this paper shows that the Furstenberg dynamical system determined by the above data is essentially unique if one is allowed to choose a suitable subsequence of Følner sets and under the additional natural assumption that the Borel sigma-algebra is the smallest sigma-algebra containing \(A\) and invariant under the action of \(G\).

The result is then discussed, stated, and proved in more general settings: without requiring \(G\) to be countable, and then even for general left amenable semigroups. It is remarked that a Furstenberg system does not need to be ergodic, and it depends in an essential way on a choice of the family of Følner sets. A partial converse to the Furstenberg correspondence is given: the multiple recurrence properties of a set in a measure-preserving system are mirrored in the recurrence properties of a set in the acting group with regard to the upper density if a Følner sequence is fixed in advance. Further, the concepts are extended to systems of finitely many functions on \(G\) instead of subsets.

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