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A survey of some norm inequalities. (English) Zbl 07352441
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Summary: We survey some classical norm inequalities of Hardy, Kallman, Kato, Kolmogorov, Landau, Littlewood, and Rota of the type

$$\|Af\|_{\mathcal{X}}^2 \leq C\|f\|_{\mathcal{X}}\|A^2f\|_{\mathcal{X}}, \quad f \in \text{dom}(A^2),$$

and recall that under exceedingly stronger hypotheses on the operator A and/or the Banach space \mathcal{X} , the optimal constant C in these inequalities diminishes from 4 (e.g., when A is the generator of a C_0 contraction semigroup on a Banach space \mathcal{X}) all the way down to 1 (e.g., when A is a symmetric operator on a Hilbert space \mathcal{H}). We also survey some results in connection with an extension of the Hardy-Littlewood inequality involving quadratic forms as initiated by Everitt.

MSC:

- 47A30 Norms (inequalities, more than one norm, etc.) of linear operators
- 34L40 Particular ordinary differential operators (Dirac, one-dimensional Schrödinger, etc.)
- 47B25 Linear symmetric and selfadjoint operators (unbounded)
- 47B44 Linear accretive operators, dissipative operators, etc.

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Keywords:

Hardy-Littlewood; Kallman-Rota inequalities; Landau-Kolmogorov inequalities

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