The book provides a compact and interesting presentation of both standard and advanced mathematical tools in harmonic analysis relevant for quantum mechanical systems described in $L^2(\mathbb{R}^n)$ spaces. The book is structured in 17 relatively short chapters. Chapter 1 fixes notation and relevant background, while chapter 2 introduces the Heisenberg-Weyl or displacements operators, together with the parity or Grossman-Royer operator and the symplectic Fourier transform, to be used in the following chapter to introduce the Wigner and scaled cross-Wigner transform. Properties and physical relevance of the latter are briefly discussed, putting into evidence the Moyal identity, suggested as a bridge between orthogonality in $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R}^{2n})$. After considering the Wigner transform of functions of special interest, the connection between phase-space functions and operators, the so-called quantization, is introduced by means of the Weyl transform in chapter 5, further considering alternative quantization schemes in the following chapters. Chapter 8 and 9 are devoted to introduce the metaplectic group and its use in studying covariance properties of the Weyl quantization. Chapter 10 introduces the Feichtinger algebra, providing an alternative to the Schwartz space in considering a Gelfand triplet and characterized by invariance properties with respect to the action of the symplectic group. The remaining chapters are devoted to the definition and study of relevant classes of operators appearing in quantum mechanics treatment, such as trace-class operators, and therefore in particular density operators, and Hilbert-Schmidt operators. The formulation is in terms of Wigner transform and Weyl symbols, but connection is made to the standard Hilbert space treatment. Crucial properties for the quantum description, such as compliance with the uncertainty principle, positivity and separability of states are considered in detail. These properties are studied in detail for the case of Gaussian states, that allow for a much richer characterization. In particular, the quantum uncertainty principle is analyzed along new interesting perspectives building on symplectic topology.

Overall the book provides a dense but clear and well structured introduction to different topics in quantum harmonic analysis, including both known fundamental results and more recent developments by the author and collaborators, in particular with reference to geometric formulations of the uncertainty principle for states in $L^2(\mathbb{R}^n)$ spaces exploiting the symplectic structure. The publication is further enriched by an informative introduction and an extensive bibliography drawing from both mathematical and physical literature.

Despite the mathematical style of the exposition, the motivation from and connection to quantum physics is always kept in the background, especially in view of foundational issues. The book can certainly be of interest to both mathematics and physics researchers and graduate students, having a good background in the formalism of quantum mechanics.

Reviewer: Bassano Vacchini (Milano)

MSC:

- 81-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to quantum theory
- 42-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to harmonic analysis on Euclidean spaces
- 81S30 Phase-space methods including Wigner distributions, etc. applied to problems in quantum mechanics
- 81P16 Quantum state spaces, operational and probabilistic concepts
- 81P40 Quantum coherence, entanglement, quantum correlations
- 42B35 Function spaces arising in harmonic analysis
- 47B90 Operator theory and harmonic analysis

Keywords:

- Wigner function; Weyl transform; symplectic group; metaplectic group; Heisenberg group; quantization; positivity; Feichtinger algebra; Gromov’s theorem