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Some properties of one-point extensions. (English) [Zbl 07354654]
Topol. Proc. 59, 195-208 (2022)

All topological spaces in this article are assumed to be Tychonoff. The set-theoretic framework is ZFC. A one-point extension of a Tychonoff space \( X \) is a Tychonoff space \( X_p \) such that there exists a homeomorphic embedding \( \pi: X \to X_p \) such that \( \pi(X) \) is dense in \( X_p \), and \( X_p \setminus \pi(X) \) is a singleton; usually, the unique point of \( X_p \setminus \pi(X) \) is denoted by \( p \), and \( X \) is identified with \( \pi(X) \); thus, \( X_p = X \cup \{p\} \) and \( X \) is a dense subspace of the Tychonoff space \( X_p \). Given a point \( p \in \beta X \setminus X_p \), the subspace \( X_p \) denotes the subspace \( \beta X \setminus X_p \) of \( \beta X \). The authors obtain the following results. A Tychonoff space \( X \) is realcompact if and only if, for every \( p \in \beta X \setminus X \), the set \( \{p\} \) is of type \( G_\delta \) in \( X_p \). A Tychonoff space \( X \) is Lindelöf (respectively, not pseudocompact) if and only if, for every \( p \in \beta X \setminus X \), the set \( \{p\} \) is of type \( G_\delta \) in \( X_p \). If a Tychonoff space \( X \) is not pseudocompact, then there exists a point \( p \in \beta X \setminus X \) such that \( X_p \) is not Fréchet-Urysohn at \( p \) (that is, there exists a set \( A \subseteq X \) such that \( p \) is an accumulation point of \( A \) but no sequence of points of \( X \) converges to \( p \) in \( X_p \)). If \( X \) is a Tychonoff space, and \( p \in \beta X \setminus X \), then \( p \) does not have a countable base of neighborhoods in \( X_p \).

A subset \( B \) of a Tychonoff space \( X \) is called \( \textit{bounded} \) if, for every continuous function \( f: X \to \mathbb{R} \), the set \( f(B) \) is bounded in \( \mathbb{R} \). The authors prove that if \( X \) is a Tychonoff space such that no infinite closed discrete subspace of \( X \) is bounded in \( X \), then, for an arbitrary \( p \in \beta X \setminus X \), the space \( X_p \) is not Fréchet-Urysohn at \( p \) because no sequence of points of \( X \) converges to \( p \) in \( \beta X \). Next, the authors consider a locally compact non-compact Hausdorff space \( Y \) such that \( \beta Y \) is a Tychonoff compactum, and \( \beta Y \setminus Y \) is a singleton. Since every one-point extension \( Y_p \) of \( Y \) can be identified with \( \beta Y \), it holds that \( Y_p \) is Fréchet-Urysohn at \( p \). The authors inform that a description of such a space \( Y \) can be found in Chapter IV. 5 in [A. V. Arhangel’skiı, Topological function spaces. Dordrecht etc.: Kluwer Academic Publishers (1992; Zbl 0758.46026)].

The authors show that there exists an Isbell-Mrówka \( \Psi \)-space \( X \) such that every one-point extension \( X_p \) of \( X \) is of countable tightness at \( p \). The authors also ask if there exists an Isbell-Mrówka \( \Psi \)-space \( X \) such that, for some point \( p \in \beta X \setminus X \), the space \( X_p \) is of uncountable tightness at \( p \).

A space \( X \) is called \( \omega \)-\( \textit{bounded} \) if every closed separable subspace of \( X \) is compact. The authors notice that if a Tychonoff space \( X \) contains a dense \( \omega \)-bounded subspace, then every one-point extension \( X_p \) of \( X \) is of uncountable tightness at \( p \). The authors show that there exists a subspace \( X' \) of the Cantor cube \( 2^{2^\omega} \) such that \( X' \) does not contain any dense \( \omega \)-bounded subspace but every one-point extension \( X_p \) of \( X \) is of uncountable tightness at \( p \). Other results on one-point extensions are also included in the article and several open problems are posed, some of them concern Q-set spaces. Let us recall that a Q-set space is an uncountable topological space \( X \) which is not \( \sigma \)-discrete but every subset of \( X \) is a \( G_\delta \)-set. Among the questions on Q-set spaces, being one of the motivations to conduct this research, there are the following: Under what conditions on a Tychonoff Q-set space \( X \) is a one-point extension \( X_p \) of \( X \) a Q-set space? Is the paracompact Q-set space constructed in [Z. T. Balogh, Proc. Am. Math. Soc. 126, No. 6, 1827–1833 (1998; Zbl 0897.54017)] Lindelöf?

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MSC:

- 54D35 Extensions of spaces (compactifications, supercompactifications, completions, etc.)
- 54B40 Remainders in general topology
- 54D20 Noncompact covering properties (paracompact, Lindelöf, etc.)

Keywords:

- one-point extension; Čech-Stone compactification; Lindelöf space; character; Fréchet-Urysohn property; \( G_\delta \)-set; pseudocompactness; Q-set space

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