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Unlikely intersections between isogeny orbits and curves. (English) Zbl 07367662
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Summary: Fix an abelian variety A_0 and a non-isotrivial abelian scheme over a smooth irreducible curve, both defined over the algebraic numbers. Consider the union of all images of translates of a fixed finite-rank subgroup of A_0 , also defined over the algebraic numbers, by abelian subvarieties of A_0 of codimension at least k under all isogenies between A_0 and some fiber of the abelian scheme. We characterize the curves inside the abelian scheme which are defined over the algebraic numbers, dominate the base curve and potentially intersect this set in infinitely many points. Our proof follows the Pila-Zannier strategy.

MSC:

11G18 Arithmetic aspects of modular and Shimura varieties
11G50 Heights
11U09 Model theory (number-theoretic aspects)
14G40 Arithmetic varieties and schemes; Arakelov theory; heights
14K02 Isogeny

Cited in **2** Documents

Keywords:

unlikely intersections; isogeny; abelian scheme; André-Pink-Zannier conjecture

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