The present paper provides an answer to the following enumeration problem: given a finite field $\mathbb{F}_q$, $q > 2$, to find the number $c_k$ of pairs of plane cubic curves, without common component, that intersect in exactly $k$ rational points in $\mathbb{P}^2(\mathbb{F}_q)$.

The value of $c_k$ for $k = 0, \ldots, 9$ (k=9 is the maximum possible value: theorem of Bézout) comes given by Theorem 1.3. Curiously enough $c_k$ is always a polynomial of degree 20 in $q$, except for $k = 8$; in this case the degree is 19. The main coefficient of that polynomial is the proportion of elements of $S_q$ with $k$ fixed points, which follows from a result of A. Entin ("Monodromy of hyperplane sections of curves and decomposition statistics over finite fields", Int. Math. Res. Not. IMRN. 2021, No. 14, 10409–10441 (2021; doi:10.1093/imrn/rnz120)].

Theorem 1.3 is proved using results of Coding Theory, concretely the weight enumerator of projective Reed-Muller codes.

Section 2 and 3 summarizes some notions about error-correcting codes defined over $\mathbb{F}_q$, in particular the weight enumerator of Reed-Muller codes and their dual and translates the original problem about cubic curves to the language of weight enumerator of projective Reed-Muller codes.

Sections 4 and 5 prove theorems analogous to Theorem 1.3 for the intersection of two conics and a conic and a cubic respectively. Then Section 7 provides the proof of Theorem 1.8.

Finally Section 8 discusses the possibility of similar results in the general problem for curves $f, g$ of degrees $d = d^o(f)$, $e = d^o(g)$.

Reviewer: Juan Tena Ayuso (Valladolid)

MSC:

14G15 Finite ground fields in algebraic geometry
11G20 Curves over finite and local fields
94B27 Geometric methods (including applications of algebraic geometry) applied to coding theory
14G50 Applications to coding theory and cryptography of arithmetic geometry
14N10 Enumerative problems (combinatorial problems) in algebraic geometry

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References:
