Feng, Ziqin; Nukala, Naga Chandra Padmini

Sub-posets in \( \omega^\omega \) and the strong Pytkeev* property. (English) [Zbl 07387388]

Topology Appl. 300, Article ID 107750, 9 p. (2021)

A partially ordered set \( Q \) is a Tukey quotient of another partially ordered set \( P \) iff there is a map \( f : P \rightarrow Q \) which maps cofinal sets of \( P \) to cofinal sets of \( Q \), here a subset \( R \) of \( P \) is cofinal iff for every \( p \in P \) there is an \( r \in R \) with \( p < r \). The Tukey order is then an order on partially ordered sets induced by the relation of being a quotient, that is, \( P \leq_T Q \) iff \( Q \) is a Tukey quotient of \( P \). Tukey classes are equivalence classes of the Tukey order. The present work uses now the Tukey order to compare partially ordered subsets of the partially ordered set \( \{0,1\}^\omega \) (product order of \( 0 < 1 \) for a countable product) and constructs a \( 2^{\omega^\omega} \) sized antichain with respect to the Tukey ordering consisting of partially ordered subsets of \( \{0,1\}^\omega \). Furthermore, partially ordered subsets of \( \omega^\omega \) are investigated.

In particular, the authors solve two recent open questions posted in the paper [J. C. Ferrando et al., Topology Appl. 208, 30–39 (2016; Zbl 1357.54017)]: First, any topological group with a \( \Sigma_2^1 \) base admits a \( \omega^\omega \)-base; second any separable metric space \( M \) is Polish iff \( K(M) \) is Tukey reducible to \( \Sigma \) for any unbounded and boundedly-complete proper partially ordered subset \( \Sigma \) of \( \omega^\omega \).

Furthermore, the authors study the strong Pytkeev* property. They provide a sufficient condition for the strong Pytkeev* property. They also investigate notions derived from a partially ordered set \( P \) meeting below specifications. Recall that metrisable means that one can define a metric on the space which generates the given topology. A second countable topology is a topological space which has a countable base of the topology. Compact sets are sets satisfying that every cover of open sets of the given set contains a finite subcover of this set. A \( P \)-base is defined locally: If the partially ordered set \( P \) is a Tukey quotient of some base of the neighbourhood of \( x \), then one says that the point \( x \) has a \( P \)-base and the whole space \( X \) has a \( P \)-base, if every \( x \in X \) has a \( P \)-base. The authors extend Theorem 1.2 of [T. Banakh, J. Kakol and J.P. Schürz, \( \omega^\omega \)-base and infinite-dimensional compact sets in locally convex spaces", to appear in Rev. Mat. Complut. (doi:10.1007/s13163-021-00397-9), Preprint: arXiv:2007.04420] by showing that each uncountably-dimensional locally convex space with a \( P \)-base contains an infinite-dimensional metrizable compact subspace if \( P \) is a directed set equipped with a second-countable topology in which every convergent sequence in \( P \) is bounded.

Reviewer: Frank Stephan (Singapore)

MSC:

54D70 Base properties of topological spaces
06A06 Partial orders, general
46B50 Compactness in Banach (or normed) spaces

Keywords:

Tukey order; strong Pytkeev* property; \( \omega^\omega \)-base; \( K(M) \)-base; \( P \)-base; locally convex space (lcs); partially ordered sets (posets); function spaces

Full Text: DOI

References:

[1] Banakh, T., Topological spaces with an \( (\omega^\omega\ \omega^\omega) \)-base, Diss. Math., 538, 141 (2019), MR3942223 · Zbl 1470.54015
[4] Banakh, T.; Leiderman, A., \( (\omega^\omega\ \omega^\omega) \)-dominated function spaces and \( (\omega^\omega\ \omega^\omega) \)-bases in free objects of topological algebra, Topol. Appl., 241, 203-241 (2018) · Zbl 1397.54036