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On the arithmetic of power monoids and sumsets in cyclic groups. (English) Zbl 07390393 Pac. J. Math. 312, No. 2, 279-308 (2021)

Summary: Let $H$ be a multiplicatively written monoid with identity $1_H$ (in particular, a group), and denote by $P_{\text{fin},x}(H)$ the monoid obtained by endowing the collection of all finite subsets of $H$ containing a unit with the operation of setwise multiplication $(X, Y) \mapsto \{xy : x \in X, y \in Y\}$. We study fundamental features of the arithmetic of this and related structures, with a focus on the submonoid, $P_{\text{fin},1}(H)$, of $P_{\text{fin},x}(H)$ consisting of all finite subsets of $H$ containing the identity.

Among others, we establish that $P_{\text{fin},1}(H)$ is atomic (i.e., each nonunit is a product of atoms) if and only if $1_H \neq x^2 \neq x$ for every $x \in H \setminus \{1_H\}$. Then we prove that $P_{\text{fin},1}(H)$ is BF (i.e., it is atomic and every element has factorizations of bounded length) if and only if $H$ is torsion-free; and we show how to transfer these conclusions from $P_{\text{fin},1}(H)$ to $P_{\text{fin},x}(H)$ through the machinery of equimorphisms.

Next, we introduce a suitable notion of “minimal factorization” (and investigate its behavior with respect to equimorphisms) to account for the fact that monoids may have nontrivial idempotents, in which case standard definitions from factorization theory degenerate. Accordingly, we obtain necessary and sufficient conditions for $P_{\text{fin},x}(H)$ to be BmF (meaning that each nonunit has at least one minimal factorization and all such factorizations are bounded in length); and for $P_{\text{fin},1}(H)$ to be BmF, HmF (i.e., a BmF-monoid where all the minimal factorizations of a given element have the same length), or minimally factorial (i.e., a BmF-monoid where each nonunit element has an essentially unique minimal factorization). Finally, we prove how to realize certain intervals as sets of minimal lengths in $P_{\text{fin},1}(H)$.

Many proofs come down to considering sumset decompositions in cyclic groups, so giving rise to an intriguing interplay with arithmetic combinatorics.

MSC:

11B30 Arithmetic combinatorics; higher degree uniformity

11P70 Inverse problems of additive number theory, including sumsets

20M13 Arithmetic theory of semigroups

11B13 Additive bases, including sumsets

Keywords:

BF-monoids; decompositions into atoms; irreducibles; minimal factorizations; nonunique factorization; power monoids; product sets; sumsets

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