Summary: We show that a triangulated motivic category admits categorical Thom isomorphisms for vector bundles with an additional structure if and only if the generalized motivic cohomology theory represented by the tensor unit object admits Thom classes. We also show that the stable $\mathbb{A}^1$-derived category does not admit Thom isomorphisms for oriented vector bundles and, more generally, for symplectic bundles. In order to do so we compute the first homology sheaves of the motivic sphere spectrum and show that the class in the coefficient ring of $\mathbb{A}^1$-homology corresponding to the second motivic Hopf map $\nu$ is nonzero, which provides an obstruction to the existence of a reasonable theory of Thom classes in $\mathbb{A}^1$-cohomology.

MSC:
14F42 Motivic cohomology; motivic homotopy theory
14F45 Topological properties in algebraic geometry
14F08 Derived categories of sheaves, dg categories, and related constructions in algebraic geometry
18G80 Derived categories, triangulated categories

Keywords:
Thom isomorphisms; Thom classes; $\mathbb{A}^1$-cohomology; $\mathbb{A}^1$-derived category; triangulated motivic category

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