Summary: By a result of Schur [J. Reine Angew. Math. 140 (1911), pp. 1-28], the entrywise product $M \circ N$ of two positive semidefinite matrices $M, N$ is again positive. Vybíral [Adv. Math. 368 (2020), p. 9] improved on this by showing the uniform lower bound $M \circ \overline{M} \geq E_n/n$ for all $n \times n$ real or complex correlation matrices $M$, where $E_n$ is the all-ones matrix. This was applied to settle a conjecture of Novak [J. Complexity 15 (1999), pp. 299-316] and to positive definite functions on groups. Vybíral (in his original preprint) asked if one can obtain similar uniform lower bounds for higher entrywise powers of $M$, or for $M \circ N$ when $N \neq M$. A natural third question is to ask for a tighter lower bound that does not vanish as $n \to \infty$, i.e., over infinite-dimensional Hilbert spaces.

In this note, we affirmatively answer all three questions by extending and refining Vybíral’s result to lower-bound $M \circ N$, for arbitrary complex positive semidefinite matrices $M, N$. Specifically: we provide tight lower bounds, improving on Vybíral’s bounds. Second, our proof is ‘conceptual’ (and self-contained), providing a natural interpretation of these improved bounds via tracial Cauchy-Schwarz inequalities. Third, we extend our tight lower bounds to Hilbert-Schmidt operators. As an application, we settle Open Problem 1 of Hinrichs-Krieg-Novak-Vybíral [J. Complexity 65 (2021), Paper No. 101544, 20 pp.], which yields improvements in the error bounds in certain tensor product (integration) problems.

MSC:

15A45 Miscellaneous inequalities involving matrices
15B48 Positive matrices and their generalizations; cones of matrices
47B10 Linear operators belonging to operator ideals (nuclear, $p$-summing, in the Schatten-von Neumann classes, etc.)
42A82 Positive definite functions in one variable harmonic analysis
43A35 Positive definite functions on groups, semigroups, etc.
46C05 Hilbert and pre-Hilbert spaces: geometry and topology (including spaces with semidefinite inner product)
47A63 Linear operator inequalities

Keywords:

positive semidefinite matrix; Schur product theorem; Loewner ordering; Hadamard product; tracial inequality; positive definite kernel

Full Text: DOI

References:


\ J.\ Math., in press; 1708.05197. - Zbl 1411.05263


This reference list is based on information provided by the publisher or from digital mathematics libraries. Its items are heuristically matched to zbMATH identifiers and may contain data conversion errors. It attempts to reflect the references listed in the original paper as accurately as possible without claiming the completeness or perfect precision of the matching.