Summary: Let $(R, \ast)$ be a ring with involution and let $A = M(n, R)$ be the matrix ring endowed with the $\ast$-transpose involution. We study $\text{SL}_\ast(2, A)$ and the question of Bruhat generation over commutative and non-commutative local and adèlic rings $R$. An important tool is the property of a ring being $\ast$-Euclidean. In this regard, we introduce the notion of a $\ast$-local ring $R$, prove that $A$ is $\ast$-Euclidean and explore reduction modulo the Jacobson radical for such rings. Globally, we provide an affirmative answer to the question whether a commutative adèlic ring $R$ leads towards the ring $A$ being $\ast$-Euclidean; while the non-commutative adèlic quaternions are such that $A$ is $\ast$-Euclidean and $\text{SL}_\ast$ is generated by its Bruhat elements if and only if the characteristic is 2.

MSC:

20G35 Linear algebraic groups over adeles and other rings and schemes
20F05 Generators, relations, and presentations of groups
16L30 Noncommutative local and semilocal rings, perfect rings

Keywords:

$\text{SL}_\ast$ groups; $\ast$-Euclidean rings; Bruhat generators