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Uniformly well-posed hybridized discontinuous Galerkin/hybrid mixed discretizations for Biot's consolidation model. (English) Zbl 07415276

Comput. Methods Appl. Mech. Eng. 384, Article ID 113991, 23 p. (2021)

Summary: We consider the quasi-static Biot's consolidation model in a three-field formulation with the three unknown physical quantities of interest being the displacement $\boldsymbol{u}$ of the solid matrix, the seepage velocity $\boldsymbol{v}$ of the fluid and the pore pressure p. As conservation of fluid mass is a leading physical principle in poromechanics, we preserve this property using an $\boldsymbol{H}(\mathrm{div})$-conforming ansatz for $\boldsymbol{u}$ and $\boldsymbol{v}$ together with an appropriate pressure space. This results in Stokes and Darcy stability and exact, that is, pointwise mass conservation of the discrete model.

The proposed discretization technique combines a hybridized discontinuous Galerkin method for the elasticity subproblem with a mixed method for the flow subproblem, also handled by hybridization. The latter allows for a static condensation step to eliminate the seepage velocity from the system while preserving mass conservation. The system to be solved finally only contains degrees of freedom related to $\boldsymbol{u}$ and p resulting from the hybridization process and thus provides, especially for higher-order approximations, a very cost-efficient family of physics-oriented space discretizations for poroelasticity problems. We present the construction of the discrete model, theoretical results related to its uniform well-posedness along with optimal error estimates and parameter-robust preconditioners as a key tool for developing uniformly convergent iterative solvers. Finally, the cost-efficiency of the proposed approach is illustrated in a series of numerical tests for three-dimensional test cases.

MSC:

76-XX Fluid mechanics

65-XX Numerical analysis

Keywords:

Biot's consolidation model; strongly mass conserving high-order discretizations; parameter-robust LBB stability; norm-equivalent preconditioners; hybrid discontinuous Galerkin methods; hybrid mixed methods

Software:

NGSolve; Netgen; poroelasticity

Full Text: DOI arXiv

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