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Rational dynamical systems, $S$-units, and $D$-finite power series. (English) Zbl 07419509
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Summary: Let $K$ be an algebraically closed field of characteristic zero, let $G$ be a finitely generated subgroup of the multiplicative group of $K$, and let $X$ be a quasiprojective variety defined over $K$. We consider $K$-valued sequences of the form $a_n := f(\varphi^n(x_0))$, where $\varphi : X \to X$ and $f : X \to \mathbb{P}^1$ are rational maps defined over $K$ and $x_0 \in X$ is a point whose forward orbit avoids the indeterminacy loci of $\varphi$ and $f$. Many classical sequences from number theory and algebraic combinatorics fall under this dynamical framework, and we show that the set of $n$ for which $a_n \in G$ is a finite union of arithmetic progressions along with a set of upper Banach density zero. In addition, we show that if $a_n \in G$ for every $n$ and $X$ is irreducible and the $\varphi$ orbit of $x$ is Zariski dense in $X$ then there is a multiplicative torus $G^d_m$ and maps $\Psi : G^d_m \to G^d_m$ and $g : G^d_m \to G_m$ such that $a_n = (g \circ \Psi^n)(y)$ for some $y \in G^d_m$. We then obtain results about the coefficients of $D$-finite power series using these facts.

MSC:

37P55 Arithmetic dynamics on general algebraic varieties
12H05 Differential algebra
14E05 Rational and birational maps

Keywords:

$S$-units; $D$-finite series; arithmetic dynamics; algebraic groups; orbits