Around spin Hurwitz numbers. (English) Zbl 07423102

Summary: We present a review of the spin Hurwitz numbers, which count the ramified coverings with spin structures. It is known that they are related to the characters of the Sergeev group and to the \( Q \) Schur functions. This allows one to put the whole story into the context of matrix models and integrable hierarchies. The generating functions of the spin Hurwitz numbers \( \tau^\pm \) are hypergeometric \( \tau \)-functions of the BKP integrable hierarchy; we present their fermionic realization. The cut-and-join equation in the form of a heat equation is written down. We explain, how a special \( d \)-soliton \( \tau \)-functions of KdV and Veselov-Novikov hierarchies generate the spin Hurwitz numbers \( H^\pm(\Gamma^d,\Delta) \). We present the well-known Kontsevich matrix integral as the BKP \( \tau \)-function in the form of special neutral fermion vacuum expectation values (few different ones). We also explain how to rewrite certain BKP \( \tau \)-functions (including the Kontsevich one) as the hypergeometric BKP \( \tau \)-functions using certain relations between the projective Schur functions.

MSC:
05A15 Exact enumeration problems, generating functions
37K10 Completely integrable infinite-dimensional Hamiltonian and Lagrangian systems, integration methods, integrability tests, integrable hierarchies (KdV, KP, Toda, etc.)
05E05 Symmetric functions and generalizations
14H81 Relationships between algebraic curves and physics
14N10 Enumerative problems (combinatorial problems) in algebraic geometry

Keywords:
spin Hurwitz numbers; integrable hierarchies; \( Q \) Schur functions

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