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Spectral ζ -functions and ζ -regularized functional determinants for regular Sturm-Liouville operators. (English) [Zbl 07423435](#)
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Summary: The principal aim in this paper is to employ a recently developed unified approach to the computation of traces of resolvents and ζ -functions to efficiently compute values of spectral ζ -functions at positive integers associated with regular (three-coefficient) self-adjoint Sturm-Liouville differential expressions τ . Depending on the underlying boundary conditions, we express the ζ -function values in terms of a fundamental system of solutions of $\tau y = zy$ and their expansions about the spectral point $z = 0$. Furthermore, we give the full analytic continuation of the ζ -function through a Liouville transformation and provide an explicit expression for the ζ -regularized functional determinant in terms of a particular set of this fundamental system of solutions. An array of examples illustrating the applicability of these methods is provided, including regular Schrödinger operators with zero, piecewise constant, and a linear potential on a compact interval.

MSC:

- 47A10 Spectrum, resolvent
- 47B10 Linear operators belonging to operator ideals (nuclear, p -summing, in the Schatten-von Neumann classes, etc.)
- 47G10 Integral operators
- 34B27 Green's functions for ordinary differential equations
- 34L40 Particular ordinary differential operators (Dirac, one-dimensional Schrödinger, etc.)
- 34B24 Sturm-Liouville theory

Keywords:

ζ -function; Sturm-Liouville operators; traces; (modified) Fredholm determinants; zeta regularized functional determinants

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