Summary: We provide necessary conditions on the regularity of domains for the optimal embeddings of first order (and higher order) Orlicz-Sobolev spaces into Orlicz spaces in the sense of [5], [6] (and [9]). We show that if $A(t) \leq C_0 t^p$ near infinity for some $p \geq 1$ and $W^{1,A}(\Omega) \hookrightarrow L^{A_n}(\Omega)$, then there exists a constant $C$ such that for every $x \in \Omega$ and $0 < r \leq 1$, $|B(x,r) \cap \Omega| \geq g^{-1}(r/C)^{-1}$, where $g(s) = \int_s^\infty \frac{A^{-1}(t) \ dt}{A(t)}$. In particular, if $I_A < n$, then $\Omega$ satisfies the measure density condition. Similarly, the embedding into the space of continuous functions implies a lower bound estimate which, in the case $i_A > n$, reduces to the measure density condition. A related condition that implies the measure density condition also in the critical case, is given by means of Orlicz-Poincaré inequalities.

We also establish the optimal embeddings in an Ahlfors regular metric measure space and prove that lower bound of the measure is a necessary condition for such embeddings. More generally, we derive a lower bound for the measure of $B(x,r) \cap \Omega$ under the assumption that $M^{1,A}(\Omega) \hookrightarrow L^A(\Omega)$, where $A$ is a Young function that increases more rapidly than $A$ near infinity. Using our results concerning embeddings combined with a generalization of a result of Heinonen and Koskela, we show that Orlicz-Sobolev extension domains satisfy the measure density condition. In the case of Hajłasz-Orlicz-Sobolev spaces, it follows that the measure density condition, or the validity of certain Orlicz-Poincaré inequalities, characterizes extension domains. We also extend the results of Korobenko-Maldonado-Rios [27] and Korobenko [26] by showing that the doubling condition on the measure is a necessary condition for some Orlicz-Poincaré inequalities.

MSC:

46E35 Sobolev spaces and other spaces of “smooth” functions, embedding theorems, trace theorems
46E30 Spaces of measurable functions ($L^p$-spaces, Orlicz spaces, Köthe function spaces, Lorentz spaces, rearrangement invariant spaces, ideal spaces, etc.)

Keywords:
Orlicz-Sobolev space; Orlicz-Sobolev embedding; measure density condition; metric measure space

Full Text: DOI

References:
