Cherubini, Giacomo; Wu, Han; Zábrádi, Gergely
On Kuznetsov-Bykovskii’s formula of counting prime geodesics. (English) Zbl 07463812
Math. Z. 300, No. 1, 881-928 (2022)

This impressive, but relatively technical, paper is motivated by the remarkable connection between the growth of algebraic data on number fields and the geometry of certain related “arithmetic” manifolds.

For instance, Peter Sarnak, in his 1980 Stanford PhD thesis, used the Selberg trace formula to prove the beautiful theorem quoted below.

Denote
\[ D = \{ d > 0 \mid d = 0, 1 \pmod{4}, d \text{ not a square} \}. \]

To each \( d \in D \), let \( h(d) \) denote the number of inequivalent primitive binary quadratic forms of discriminant \( d \), and let \((x_d, y_d)\) be the fundamental solution of the Pellian equation \( x^2 - dy^2 = 4 \), and let \( \epsilon_d = \frac{x_d + \sqrt{dy_d}}{2} \).

Let \( \Gamma = \text{PSL}(2, \mathbb{Z}) \) and let \( H \) denote the upper half plane.

Theorem: The lengths of the closed geodesics on \( H/\Gamma \) are the numbers \( 2 \log \epsilon_d \) with multiplicity \( h(d) \).

Since then, many mathematicians have contributed significantly to exploring this connection using the Selberg trace formula and its analogues, such as the Kuznetsov trace formula.

It’s easy to imagine that the Kuznetsov-Bykovskii formula of the paper’s title is some version of the Kuznetsov trace formula derived by Bykovskii. However, this natural guess is wrong. In fact, Kuznetsov discovered his trace formula in the 1980s but the formula was discovered earlier, in the late 1970s, published in a very obscure Russian preprint. Bykovskii’s work, based on it, dates from the 1990s.

In any case, to describe the Kuznetsov-Bykovskii formula, recall if \( \{P\} \) is a hyperbolic conjugacy class that is a power of the primitive hyperbolic class \( \{P_0\} \) then the von Mangoldt function is defined by \( \Lambda(P) = \log(NP_0) \) (if need be, it can be extended by 0 on the other classes). Here \( N \) is the norm function, defined to be \( NP_0 = \lambda^2 \), if \( P_0 \) can be represented as an \( SL(2, \mathbb{R}) \)-conjugacy class by \( \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \) with \( \lambda > 0 \).

One side of the Kuznetsov-Bykovskii formula involves the Chebyshev function, \( \Psi_\Gamma(x) = \sum_{NP \leq x} \Lambda(P) \), where the sum is over all hyperbolic conjugacy classes \( \{P\} \). Sarnak’s work, mentioned above, established a formula relating this Chebyshev function to a certain sum involving class numbers. In a recent paper, K. Soundararajan and M. P. Young [J. Reine Angew. Math. 676, 105–120 (2013; Zbl 1276.11084)] use this formula of Sarnak and some formulas of Zagier dating from the mid-1970s to derive the following result.

Theorem. (Kuznetsov-Bykovskii formula): Let \( \Gamma = \text{PSL}(2, \mathbb{Z}) \). We have
\[
\Psi_\Gamma(x) = \sum_{n \leq x^{1/2} + x^{-1/2}} \sqrt{n^2 - 4L(1, n^2 - 4)},
\]
where \( L(s, \delta) \) is a certain Zagier-Dirichlet series.

The main result of the paper under review is to generalize this theorem significantly. For example, they prove an analogue where \( \Gamma = \text{PSL}(2, \mathbb{Z}) \) is replaced by a principal congruence subgroup.

The proof, which is very technical, ultimately relies on methods to prove the Selberg trace formula due to Jacquet and Zagier from the 1980s. While their derivation, using a version of the Rankin-Selberg method, was incomplete due to some difficult technical issues, these technical problems were recently resolved by the second named author (Wu). The method that the authors use to prove their generalization of the Kuznetsov-Bykovskii formula is not merely an analogue of the Soundararajan-Young derivation mentioned above. Instead, they reformulate the problem in an adelic framework and then proceed along the lines Wu used in his earlier paper.

As an application, the authors obtain an estimate of the form \( \Phi_\Gamma(x) = x + O_{\varepsilon}(x^{C+\varepsilon}) \), for each \( \varepsilon > 0 \), where \( C < 3/4 \) is an explicitly given constant. This result generalizes estimates obtained earlier by several other mathematicians, such as Iwaniec in the 1980s and the above-mentioned K. Soundararajan and M. Young in the 2010s.
For a more precise description of their relatively technical results, as well as a detailed bibliography, please see the paper itself.

Reviewer: David Joyner (Annapolis)

**MSC:**

- 11F72 Spectral theory; trace formulas (e.g., that of Selberg)
- 11F06 Structure of modular groups and generalizations; arithmetic groups
- 11M36 Selberg zeta functions and regularized determinants; applications to spectral theory, Dirichlet series, Eisenstein series, etc. (explicit formulas)

**Keywords:**

- prime geodesic theorem; Rankin-Selberg method

**Full Text:** DOI

**References:**


[17] Luo, W., Sarnak, P.: Quantum ergodicity of eigenfunctions on \((\text{PSL}_2(\mathbb{Z})\backslash \mathbb{H})\) \& backslash mathbb H^\times \mathbb R \times 2\). Publications mathématiques de l'IHÉS 81, 207-237 (1995)


