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Sums of squares and quadratic persistence on real projective varieties. (English)

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Summary: We bound the Pythagoras number of a real projective subvariety: the smallest positive integer r such that every sum of squares of linear forms in its homogeneous coordinate ring is a sum of at most r squares. Enhancing existing methods, we exhibit three distinct upper bounds involving known invariants. In contrast, our lower bound depends on a new invariant of a projective subvariety called the quadratic persistence. Defined by projecting away from points, this numerical invariant is closely related to the linear syzygies of the variety. In addition, we classify the projective subvarieties of maximal and almost-maximal quadratic persistence, and determine their Pythagoras numbers.

MSC:

14P05 Real algebraic sets

52A99 General convexity

13D02 Syzygies, resolutions, complexes and commutative rings

Keywords:

convex algebraic geometry; sums of squares; Pythagoras number; linear syzygies

Software:

SDPLR

Full Text: [DOI](#) [arXiv](#)

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