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Cardinal-indexed classifying spaces for families of subgroups of any topological group.

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Summary: For G a topological group, existence theorems by Milnor (1956), Gelfand-Fuks (1968), and Segal (1975) of classifying spaces for principal G -bundles are generalized to G -spaces with torsion. Namely, any G -space approximately covered by tubes (a generalization of local trivialization) is the pullback of a universal space indexed by the orbit types of tubes and cardinality of the cover. For G a Lie group, via a metric model we generalize the corresponding uniqueness theorem by Palais (1960) and Bredon (1972) for compact G . Namely, the G -homeomorphism types of proper G -spaces over a metric space correspond to stratified-homotopy classes of orbit classifying maps.

The former existence result is enabled by Segal's clever but esoteric use of non-Hausdorff spaces. The latter uniqueness result is enabled by our own development of equivariant ANR theory for noncompact Lie G . Applications include the existence part of classification for unstructured fiber bundles with locally compact Hausdorff fiber and with locally connected base or fiber, as well as for equivariant principal bundles which in certain cases via other models is due to Lashof-May (1986) and to Lück-Urbe (2014). From a categorical perspective, our general model $E_{\mathcal{F}}^c G$ is a final object inspired by the formulation of the Baum-Connes conjecture (1994).

MSC:

- 55Pxx Homotopy theory
- 57Sxx Topological transformation groups
- 54H11 Topological groups (topological aspects)
- 55R15 Classification of fiber spaces or bundles in algebraic topology
- 58A35 Stratified sets
- 54C55 Absolute neighborhood extensor, absolute extensor, absolute neighborhood retract (ANR), absolute retract spaces (general properties)
- 57S20 Noncompact Lie groups of transformations

Keywords:

classifying space; transformation group; stratified space; equivariant absolute neighborhood retract; non-compact Lie group

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References:

- [1] Abels, H., A universal proper G -space, *Math. Z.*, 159, 2, 143-158 (1978)
- [2] Ageev, S. M., Универсальные G -пространства Пале и изовариантные абсолютные экстензоры, *Математический Сборник*, 203, 6, 3-34 (2012)
- [3] Aleksandrov, P., Diskrete Räume, *Математический Сборник*, 2, 44, 501-519 (1937)
- [4] Aleksandrov, P. S.; Hopf, H., *Topologie*, Grundlehren der Mathematischen Wissenschaften, vol. 45 (1935), Springer: Springer Berlin
- [5] Antonyan, N.; Antonyan, S.; Rodríguez-Medina, L., Linearization of proper group actions, *Topol. Appl.*, 156, 11, 1946-1956 (2009)
- [6] Antonyan, N.; Antonyan, S.; Varela-Velasco, R., Universal G -spaces for proper actions of locally compact groups, *Topol. Appl.*, 159, 4, 1159-1168 (2012)
- [7] Antonyan, S., Эквивариантное обобщение теоремы Дугунджи, *Математические Заметки*, 38, 4, 608-616 (1985)
- [8] Antonyan, S., Equivariant embeddings into G -ARs, *Glas. Mat. Ser. III*, 22, 2, 503-533 (1987)
- [9] Antonyan, S., Extensorial properties of orbit spaces of proper group actions, *Topol. Appl.*, 98, 1-3, 35-46 (1999)
- [10] Antonyan, S., Orbit spaces and unions of equivariant absolute neighborhood extensors, *Topol. Appl.*, 146/147, 289-315 (2005)
- [11] Antonyan, S.; de Neymet, S., Invariant pseudometrics on Palais proper G -spaces, *Acta Math. Hung.*, 98, 1-2, 59-69 (2003)

- [12] Arens, R., Topologies for homeomorphism groups, *Am. J. Math.*, 68, 593-610 (1946)
- [13] Arens, R.; Eells, J., On embedding uniform and topological spaces, *Pac. J. Math.*, 6, 397-403 (1956)
- [14] Banakh, T. O., О топологии конструкции Милнора универсального G -расслоения, *Сибирский Математический Журнал*, 33, 1, 16-25 (1992)
- [15] Baum, P.; Connes, A.; Higson, N., Classifying space for proper actions and K -theory of group C^* -algebras, *(C^* -Algebras 1943-1993. C^* -Algebras 1943-1993, San Antonio, TX, 1993. C^* -Algebras 1943-1993. C^* -Algebras 1943-1993, San Antonio, TX, 1993, Contemp. Math., vol. 167 (1994), Amer. Math. Soc.: Amer. Math. Soc. Providence, RI), 240-291*
- [16] Biller, H., Characterizations of proper actions, *Math. Proc. Camb. Philos. Soc.*, 136, 2, 429-439 (2004)
- [17] Björner, A., Posets, regular CW complexes and Bruhat order, *Eur. J. Comb.*, 5, 1, 7-16 (1984)
- [18] Bourbaki, N., Structures topologiques, *Éléments de mathématique: Topologie générale*, vol. I (1961), *Actualités Sci. Indust. #1142: Actualités Sci. Indust. #1142 Paris*
- [19] Bourbaki, N., Ensembles convexes et espaces localement convexes, *Éléments de mathématique: Espaces vectoriels topologiques*, vol. II (1981), *Masson: Masson Paris*
- [20] Bredon, G. E., *Introduction to Compact Transformation Groups*, *Pure and Applied Mathematics*, vol. 46 (1972), *Academic Press: Academic Press New York and London*
- [21] Cartan, E., *La théorie des groupes finis et continus et l'Analysis situs*, *Mémorial des Sciences Mathématiques*, vol. 42 (1930), *Gauthier-Villars: Gauthier-Villars Paris*
- [22] Čech, E., On bicomact spaces, *Ann. Math.*, 38, 4, 823-844 (1937)
- [23] Cianci, N.; Ottina, M., Fiber bundles over Alexandroff spaces (2020)
- [24] Crowell, R. H., Invertible isotopies, *Proc. Am. Math. Soc.*, 14, 658-664 (1963)
- [25] tom Dieck, T., Klassifikation numerierbarer Bündel, *Arch. Math. (Basel)*, 17, 395-399 (1966)
- [26] tom Dieck, T., Orbittypen und äquivariante Homologie, I, *Arch. Math. (Basel)*, 23, 307-317 (1972)
- [27] tom Dieck, T., *Transformation Groups*, *De Gruyter Studies in Mathematics*, vol. 8 (1987), *Walter de Gruyter: Walter de Gruyter Berlin*
- [28] Dietze, A.; Schaps, M., Determining subgroups of a given finite index in a finitely presented group, *Can. J. Math.*, 26, 769-782 (1974)
- [29] Dieudonné, J., Une généralisation des espaces compacts, *J. Math. Pures Appl. (9)*, 23, 65-76 (1944)
- [30] Dold, A., Partitions of unity in the theory of fibrations, *Ann. Math.*, 78, 223-255 (1963)
- [31] Dowker, C., On countably paracompact spaces, *Can. J. Math.*, 3, 219-224 (1951)
- [32] Dugundji, J., An extension of Tietze's theorem, *Pac. J. Math.*, 1, 353-367 (1951)
- [33] Ehresmann, C.; Feldbau, J., Sur les propriétés d'homotopie des espaces fibrés, *C. R. Acad. Sci. Paris*, 212, 945-948 (1941)
- [34] Elmendorf, A. D., Systems of fixed point sets, *Trans. Am. Math. Soc.*, 277, 1, 275-284 (1983)
- [35] Engelking, R., *General Topology*, *Sigma Series in Pure Mathematics*, vol. 6 (1989), *Heldermann Verlag: Heldermann Verlag Berlin*, Translated from the Polish
- [36] Fox, R., On topologies for function spaces, *Bull. Am. Math. Soc.*, 51, 429-432 (1945)
- [37] Fréchet, M., Sur la notion de voisinage dans les ensembles abstraits, *Bull. Sci. Math.*, 42, 138-156 (1918)
- [38] Fritsch, R.; Golasinski, M., Topological, simplicial and categorical joins, *Arch. Math. (Basel)*, 82, 5, 468-480 (2004)
- [39] Gelfand, I. M.; Fuks, D. V., О классифицирующих пространствах для главных расслоений с хаусдорфовыми базами, *Доклады Академии Наук СССР*, 181, 3, 515-518 (1968)
- [40] Goresky, M.; MacPherson, R., *Stratified Morse Theory*, *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*, vol. 14 (1988), *Springer-Verlag: Springer-Verlag Berlin*
- [41] Guillou, B. J.; May, J. P.; Merling, M., Categorical models for equivariant classifying spaces, *Algebraic Geom. Topol.*, 17, 5, 2565-2602 (2017)
- [42] Haver, W. E., Locally contractible spaces that are absolute neighborhood retracts, *Proc. Am. Math. Soc.*, 40, 280-284 (1973)
- [43] Hofmann, K. H.; Morris, S. A., *The Structure of Compact Groups*, *Studies in Mathematics*, vol. 25 (2006), *Walter de Gruyter: Walter de Gruyter Berlin*
- [44] Holm, P., The microbundle representation theorem, *Acta Math.*, 117, 191-213 (1967)
- [45] Hu, S.-T., *Theory of Retracts* (1965), *Wayne State University Press: Wayne State University Press Detroit*
- [46] Hughes, C. B., Stratified path spaces and fibrations, *Proc. R. Soc. Edinb., Sect. A*, 129, 2, 351-384 (1999)
- [47] Husemöller, D., *Fibre Bundles* (1966), *McGraw-Hill Book Co: McGraw-Hill Book Co NewYork-London-Sydney*
- [48] James, I.; Segal, G., On equivariant homotopy theory, (*Topology Symposium. Topology Symposium, Siegen 1979. Topology Symposium. Topology Symposium, Siegen 1979, Lecture Notes in Math.*, vol. 788 (1980), *Springer*), 316-330
- [49] Khan, Q., Countable approximation of topological G -manifolds, III: arbitrary Lie groups G , *N.Y. J. Math.*, 27, 1554-1579 (2021)
- [50] Kolmogorov, A. N., Zur Normierbarkeit eines allgemeinen topologischen linearen Raumes, *Stud. Math.*, 5, 29-33 (1934)

- [51] Lashof, R., Equivariant bundles, *Ill. J. Math.*, 26, 2, 257-271 (1982)
- [52] Lashof, R.; May, J. P., Generalized equivariant bundles, *Bull. Soc. Math. Belg. A*, 38, 265-271 (1986)
- [53] Lashof, R.; Rothenberg, M., G-smoothing theory, (*Algebraic and Geometric Topology. Algebraic and Geometric Topology, Proc. Sympos. Pure Math.*, vol. 32 (1978), American Mathematical Society: American Mathematical Society Providence), 211-266
- [54] Lefschetz, S., *Topics in Topology, Annals of Mathematics Studies*, vol. 10 (1942), Princeton University Press: Princeton University Press Princeton
- [55] Lindelöf, E., Sur quelques points de la théorie des ensembles, *C. R. Acad. Sci. Paris*, 137, 697-700 (1904)
- [56] Lück, W., Survey on classifying spaces for families of subgroups, (*Infinite Groups: Geometric, Combinatorial and Dynamical Aspects. Infinite Groups: Geometric, Combinatorial and Dynamical Aspects, Progr. Math.*, vol. 248 (2005), Birkhäuser: Birkhäuser Basel), 269-322
- [57] Lück, W.; Uribe, B., Equivariant principal bundles and their classifying spaces, *Algebraic Geom. Topol.*, 14, 4, 1925-1995 (2014)
- [58] Lundell, A. T.; Weingram, S., *The Topology of CW Complexes, The University Series in Higher Mathematics* (1969), Van Nostrand Reinhold Co.: Van Nostrand Reinhold Co. New York
- [59] Lurie, J., *Higher Algebra* (2017)
- [60] MacLane, S., *Categories for the Working Mathematician, Graduate Texts in Mathematics*, vol. 5 (1971), Springer-Verlag: Springer-Verlag New York-Berlin
- [61] McCord, M. C., Singular homology groups and homotopy groups of finite topological spaces, *Duke Math. J.*, 33, 465-474 (1966)
- [62] Michael, E., A note on paracompact spaces, *Proc. Am. Math. Soc.*, 4, 831-838 (1953)
- [63] Milnor, J. W., Construction of universal bundles, II, *Ann. Math.*, 63, 430-436 (1956)
- [64] Milnor, J. W., *Lectures on Characteristic Classes* (1957), Princeton University, Notes by James Stasheff
- [65] Milnor, J. W., Microbundles, *Topology*, 3, s1, 53-80 (1964)
- [66] Milnor, J. W.; Stasheff, J. D., *Characteristic Classes, Annals of Mathematics Studies*, vol. 76 (1974), Princeton University Press/University of Tokyo Press: Princeton University Press/University of Tokyo Press Princeton, N. J./Tokyo
- [67] Montgomery, D.; Zippin, L., *Topological Transformation Groups* (1955), Interscience Publishers: Interscience Publishers New York-London
- [68] Munkres, J. R., *Topology* (2000), Prentice Hall: Prentice Hall Upper Saddle River
- [69] Palais, R. S., The Classification of G-Spaces, *Memoirs of the AMS*, vol. 36 (1960), American Mathematical Society: American Mathematical Society Providence
- [70] Palais, R. S., On the existence of slices for actions of non-compact Lie groups, *Ann. Math.*, 73, 295-323 (1961)
- [71] Pontryagin, L. S., *Непрерывные Группы* (1954), Государственное Издательство Техничко-Теоретической Литературы: Государственное Издательство Техничко-Теоретической Литературы Moscow
- [72] Segal, G., Классифицирующее пространство топологической группы в смысле Гельфанда-Фукса, *Функциональный Анализ и его Приложения*, 9, 2, 48-50 (1975)
- [73] Sierpiński, W., *General Topology, Mathematical Expositions*, vol. 7 (1952), University of Toronto Press: University of Toronto Press Toronto, Translated from the Polish
- [74] Steen, L. A.; Seebach, J. A., *Counterexamples in Topology* (1978), Springer Verlag: Springer Verlag New York and Heidelberg
- [75] Steenrod, N. E., *The Topology of Fibre Bundles, Princeton Mathematical Series*, vol. 14 (1951), Princeton University Press: Princeton University Press Princeton
- [76] Steenrod, N. E., A convenient category of topological spaces, *Mich. Math. J.*, 14, 133-152 (1967)
- [77] Stone, A. H., Paracompactness and product spaces, *Bull. Am. Math. Soc.*, 54, 977-982 (1948)
- [78] Tietze, H., Über Funktionen, die auf einer abgeschlossenen Menge stetig sind, *J. Reine Angew. Math.*, 145, 9-14 (1915)
- [79] Tikhonov, A. N., Über die topologische Erweiterung von Räumen, *Math. Ann.*, 102, 1, 544-561 (1930)
- [80] Urysohn, P., Über die Mächtigkeit der zusammenhängenden Mengen, *Math. Ann.*, 94, 1, 262-295 (1925)
- [81] Weinberger, S., *The Topological Classification of Stratified Spaces, Chicago Lectures in Mathematics* (1994), University of Chicago Press: University of Chicago Press Chicago
- [82] Whitehead, J. H.C., Simplicial spaces, nuclei and m-groups, *Proc. Lond. Math. Soc.*, 45, 4, 243-327 (1939)
- [83] Zhang, L.-L.; Antonyan, S.; Antonyan, N., Universal G-spaces for proper free actions, *Topol. Appl.*, 264, 336-351 (2019)

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