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**Homogeneous spaces, algebraic $K$-theory and cohomological dimension of fields.** (English)

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Let $K$ be a field of characteristic $\operatorname{char}(K)$, $K_{\mathrm{sep}}$ a separable closure of $K$, $\mathcal{G}_K$ the absolute Galois group of $K$ (defined to be the Galois group $\mathcal{G}(K_{\mathrm{sep}}/K)$), and for each prime number $\ell$, let $\operatorname{cd}_\ell(\mathcal{G}_K)$ be the $\ell$-cohomological dimension of $\mathcal{G}_K$. By definition, the $\ell$-cohomological dimension $\operatorname{cd}_\ell(K)$ of $F$ is equal to $\operatorname{cd}_\ell(\mathcal{G}_K)$, for every prime $\ell \neq char(K)$. It is well-known that $\operatorname{cd}_p(\mathcal{G}_F) \leq 1$ if $\operatorname{char}(F) = p > 0$ (see [*J.-P. Serre*, Cohomologie Galoisienne. 5ème éd., rév. et complété. Berlin: Springer-Verlag (1994; Zbl 0812.12002)]). Then $\operatorname{cd}_p(K)$ is defined to be the smallest integer $i$ such that $[K\colon K^p] \leq p^i$ and the Kato-Milne cohomology group $H_p^{i+1}(L)$ (for its definition, see [*J. S. Milne*, Ann. Sci. Éc. Norm. Supér. (4) 9, 171–201 (1976; Zbl 0334.14010)]) is trivial, for every finite extension $L$ of $K$; $\operatorname{cd}_p(K)$ is infinity if $i$ does not exist. The cohomological dimension $\operatorname{cd}(K)$ of $K$ is the supremum of all the $\operatorname{cd}_\ell(K)$ when $\ell$ runs across the set of prime numbers. It follows from the Bloch-Kato conjecture (proved by *V. Voevodsky* [Ann. Math. (2) 174, No. 1, 401–438 (2011; Zbl 1236.14026)], *J. Riou* [Astérisque 361, 421–463, Exp. No. 1073 (2014; Zbl 1366.19001)], and further references there) that $\operatorname{cd}(K) \leq n$ if and only if $K$ satisfies condition $C_0^n$ introduced in: [*K. Kato* and *T. Kuzumaki*, J. Number Theory 24, 229–244 (1986; Zbl 0608.12029)].

The research presented in the paper under review is motivated by the influence of some diophantine properties of fields on their cohomological dimension. Perhaps the earliest result of this kind states that any finitely-generated extension $F$ of an algebraically closed field $F_0$ of transcendence degree $d$ is an $C_d$-field with $\operatorname{cd}(\mathcal{G}_F) = d$ (see [*S. Lang*, Ann. Math. (2) 55, 373–390 (1952; Zbl 0046.26202)], and Ch. II, Proposition 11 of [Serre, loc. cit.]). When $K$ is a field with $\operatorname{char}(K) = 0$, the condition that $\operatorname{cd}(K) \leq q \leq 2$ is characterized by the surjectivity of: norm mappings of finite separable extensions $L'/L$ of $K$, for $q = 1$; reduced norm mappings of finite-dimensional central simple algebras over finite extensions of $K$, for $q = 2$. This ensures that $\operatorname{cd}(K) \leq q$ in case $K$ is a $C_q$-field (for $q = 1$ and $q = 2$, see Ch. II, 3.1, of [Serre, loc. cit.], and Theorem 24.8, Corollary 24.9 in: [*A. A. Suslin*, J. Sov. Math. 30, 2556–2611 (1985; Zbl 0566.12016); *A. A. Suslin*, in: Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat. 25, 115–207 (1984; Zbl 0558.12013)], respectively). The latter result has been generalized to the case where $\operatorname{char}(K) \neq 0$ (see Theorem 7 in: [*P. Gille*, *K*-Theory 21, No. 1, 57–100 (2000; Zbl 0993.20031)]). It has also been proved that $\operatorname{cd}(\mathcal{G}_E) \leq q$ if $E$ is a $C_q$-field and $3 \geq q \leq 4$ [*D. Krashen* and *E. Matzri*, Proc. Am. Math. Soc. 143, No. 7, 2779–2788 (2015; Zbl 1329.12003)]. On the other hand, a number of results obtained in the course of time (including [*J. Ax*, Proc. Am. Math. Soc. 16, 1214–1221 (1965; Zbl 0142.30001); *A. S. Merkur'ev*, Math. USSR, Izv. 38, No. 1, 215–221 (1991; Zbl 0763.12003); translation from Izv. Akad. Nauk SSSR, Ser. Mat. 55, No. 1, 218–224 (1991); *J.-L. Colliot-Thélène* and *D. A. Madore*, J. Inst. Math. Jussieu 3, No. 1, 1–16 (2004; Zbl 1056.14030)]) show that neither the classical $C_m$-property nor its variant $C_q^0$ introduced by Kato and Kuzumaki (loc. cit.) characterize the cohomological dimension of fields.

The reviewed paper introduces a variant of the $C_1^q$ property and proves that, contrary to the $C_1^q$ property, they characterize the cohomological dimensions of fields. It proves that $\operatorname{cd}(K) \leq q$ if and only if, for any finite extension $L/K$ and for any homogeneous space $Z$ under a smooth linear connected algebraic group defined over $L$, the $q$-th Milnor $K$-theory group $K_q^M(L)$ is spanned by the images of the norms coming from those finite extensions of $L$ over which $Z$ has a rational point. The authors also obtain a variant of this result for imperfect fields. As explained in the text, the main theorem of the present paper unifies and significantly generalizes the above-noted results of Suslin and Gille as well as theorems due to Springer and Steinberg (see Ch. II, 2.4, in [Serre, loc. cit]), and Wittenberg (see Corollaries 5.6 and 5.8 in: [*O. Wittenberg*, Duke Math. J. 164, No. 11, 2185–2211 (2015; Zbl 1348.11037)]).

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19D45 Higher symbols, Milnor $K$-theory
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