Given Banach spaces $E$ and $F$ and a continuous convex function $\psi: [0, 1] \to \mathbb{R}$ such $\psi(0) = \psi(1) = 1$ and $\max\{1-t, 1\} \leq \psi(t) \leq 1$ the author uses $E \oplus_{\psi} F$ to denote the direct sum of $E$ and $F$ equipped with the norm $\| \cdot \|_\psi$ where $\|(x, y)\|_\psi = (\|x\| + \|y\|)/\left(\|x\| + \|y\|\right)$ if $(x, y) \neq (0, 0)$ and $\|(0, 0)\|_\psi = 0$. Given a subset $K$ of $E \oplus F$ the author defines the distance function $d_K: E \oplus F \to \mathbb{R}$ by

$$d_K(x, y) := \inf\{\|(x - u, y - v)\| : (u, v) \in K\}.$$ 

He says that a subset $K$ of $E \oplus F$ is Chebyshev if the set $\{(z, w) \in K : \|(x - z, y - w)\| = d_K(x, y)\}$ is a singleton for every $(x, y)$ in $E \oplus F$ and is biconvex if each of its $x$ and $y$ sections are convex subsets of $F$ and $E$ respectively. The author proves that for Banach spaces $E$ and $F$ with rotund duals and a Chebyshev biconvex subset $K$ of $E \oplus_{\psi} F$ the following conditions (among others) are equivalent:

(i) $K$ is convex,
(ii) $d_K$ is a convex function,
(iii) $d_K$ is upper-upper regular on $(E \oplus_{\psi} F) \setminus K$,
(iv) $d_K$ is strictly differentiable on $(E \oplus_{\psi} F) \setminus K$,
(v) $d_K$ is Gâteaux differentiable on $(E \oplus_{\psi} F) \setminus K$.

Given Banach spaces $E$, $F$ and $G$, bilinear mappings $B: E \times E \to F$ and $H: E \times E \to G$ are said to be coprime if there is $c > 0$ so that

$$\|B(x, y)\| + \|H(x, y)\| \geq c \max\{\|x\|, \|y\|\}$$

for all $(x, y)$ in $E \times E$. When $B$ and $H$ have dense range, $t_0 \in F$, $t_1 \in G$ and $c > 0$ the Bilinear Extremal Problem is to find $(x_0, y_0) \in K_{t_0} := \{(x, y) \in E \times E : \|H(x, y) - t_0\| \leq \epsilon\}$ so that

$$\|B(x_0, y_0) - t_1\| = \inf\{\|B(x, y) - t_1\| : (x, y) \in K_{t_0}\}.$$ 

The author gives various sufficient conditions on coprime bilinear mappings $B$ and $H$ with dense range in order that the Bilinear Extremal Problem has a positive solution. They include the conditions that $F$ and $G$ are reflexive and that $B$ is weakly sequentially continuous or the condition that $E$ is smooth and reflexive, $G$ is reflexive, $B$ is weakly sequentially continuous and $d_K$ has a Gâteaux derivative on $(E \oplus E) \setminus K_{t_0}$. The uniqueness of a solution to the Bilinear Extremal Problem is also discussed.

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MSC:

46G25 (Spaces of) multilinear mappings, polynomials
41A65 Abstract approximation theory (approximation in normed linear spaces and other abstract spaces)

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