Eisenbud, David

Green's conjecture: An orientation for algebraists. (English) Zbl 0792.14015

Free resolutions in commutative algebra and algebraic geometry, Proc. Conf., Sundance/UT (USA) 1990, Res. Notes Math. 2, 51-78 (1992).

[For the entire collection see Zbl 0745.00042.]

The first section of this paper leads to an algebraic conjecture generalizing Green's conjecture.

Let $S = k[x_0, \dots, x_r]$, and let $R = S/I$ be a homogeneous factor ring of S. We assume that I contains no linear forms, and the projective dimension of S/I is m. Then minimal free resolution $\mathcal{F}$ of S/I can be written as

$$0 \leftarrow S/I \leftarrow S \leftarrow S(-2)^{a_1} \oplus S(-3)^{b_1} \oplus \cdots \leftarrow S(-3)^{a_2} \oplus S(-4)^{b_2} \oplus \cdots$$
$$\leftarrow \cdots \leftarrow S(-(m+1))^{a_m} \oplus S(-(m+2))^{b_m} \oplus S(-(m+3))^{c_m} \oplus \cdots \leftarrow 0,$$

with $a_i, b_i \in \mathbb{Z}$, $a_i, b_i \geq 0$. We define the 2-linear strand of $\mathcal{F}$ to be the subcomplex

$$0 \leftarrow S/I \leftarrow S \leftarrow S(-2)^{a_1} \leftarrow S(-3)^{a_2} \leftarrow \cdots \leftarrow S(-(m+1))^{a_m} \leftarrow 0.$$

The author defines the length of the 2-linear strand to be the largest number n such that $a_n \neq 0$. He calls this n the 2-linear projective dimension and writes $2\mathrm{LP}(S/I) = n$.

If I is the ideal generated by the 2×2 minors of a generic $p \times q$ matrix, then the 2-linear strand is known to have length $\geq p + q - 3$. As a form of converse the author is lead to the following algebraic conjecture: Let k be an algebraically closed field of characteristic $\neq 2$, and let $I \subset S = k[x_0, \dots, x_r]$ be a prime ideal, containing no linear form, whose quadratic part is spanned by quadrics of rank ≤ 4. If $2\mathrm{LP}(S/I) = n$, then I contains an ideal of 2×2 minors of a 1-generic $p \times q$ matrix with $p + q - 3 = n$.

Green's conjecture, from the algebraic point of view, is just the special case of this where (a) S/I is normal (= integrally closed); (b) $\dim S/I = 2$; (c) S/I is Gorenstein; (d) degree $S/I = 2r$.

In section two the author considers the canonical ring of a non- hyperelliptic curve (= the homogeneous coordinate ring of the canonically embedded curve) and gets the geometric conjecture [*M. L. Green*, Invent. Math. 75, 85-104 (1984; Zbl 0542.14018)]:

The length of the 2-linear part of the resolution $\mathcal{F}$ of the canonical ring S/I of a curve of genus g and Clifford index c is $2\mathrm{LP}(S/I) = g - 2 - c$. The generic form there of becomes:

The free resolution of the canonical ring of a generic curve of genus g has $a_{\lfloor g/2 \rfloor}, \dots, a_{g-3} = 0, 0, \dots, 0$.

The third section of the paper surveys some approaches to Green's conjecture.

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MSC:

14H99 Curves in algebraic geometry
13D25 Complexes (MSC2000)
13D05 Homological dimension and commutative rings
14H45 Special algebraic curves and curves of low genus

Keywords:

canonical ring of a non-hyperelliptic; minimal free resolution; 2-linear projective dimension; genus; Clifford index