DiBenedetto, Emmanuele

Degenerate parabolic equations. (English) Zbl 0794.35090

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The author gives a thorough treatment of equations and systems of equations modeled on the so-called parabolic p-Laplace equation: $u_t = \operatorname{div}(|Du|^{p-2}Du)$, where $p > 1$ is a real parameter. Such equations are parabolic except where $Du = 0$ (except the special case $p = 2$, which is the classical heat equation). They degenerate if $p > 2$ and they are singular if $p < 2$. For this reason, one does not usually expect such equations to have classical solutions, and an important element of this book is the study of regularity properties of weak solutions. The corresponding theory for the analogous elliptic equations $\operatorname{div}(|Du|^{p-2}Du) = 0$ is well-known by now. In the early 1960's, Ladyzhenskaya and Ural'tseva showed that weak solutions of the elliptic equations are necessarily Hölder continuous, and Serrin showed that nonnegative solutions also satisfy a Harnack type inequality. Then, in 1968, Ural'tseva showed that the gradient Du is also Hölder for $p > 2$; the case $p < 2$ was proved by Lewis in 1981. All of these estimates were based on the studies of linear elliptic equations with bounded measurable coefficients by DeGiorgi in 1957 and Moser in 1960. Unfortunately, these elliptic techniques have easy parabolic analogs only in case $p = 2$. For the general degenerate and singular parabolic equations in this book, DiBenedetto introduces an intrinsic rescaling. To explain this concept briefly, we recall that the natural sets for studying the heat equation are cylinders of the form $Q(r) = \{|x| < r, -r^2 < t < 0\}$ and their translates. When $p \neq 2$, the author uses cylinders of the form $Q(\alpha r) = \{|x| < r, -\alpha r^2 < t < 0\}$ for a positive parameter α, which depends on the number of space dimensions, the parameter p, and some quantitative measure of the solution u in $Q(\alpha r)$. For example, the Hölder gradient estimate uses an α which depends on the maximum of the gradient over $Q(\alpha r)$. Because of this dependence, the usual geometric considerations for studying the heat equation are made more complicated. In addition, the generally different behavior of solutions to degenerate equations versus solutions to singular equations requires the two situations to be handled somewhat differently. This book details the author's investigation into such problems. The clear exposition style is worth noting because of the necessarily technical nature of the subject matter. DiBenedetto does an excellent job of leading the reader through what could be treacherous waters.

Reviewer: G.M.Lieberman (Ames)

MSC:

35K65	Degenerate parabolic equations
35K20	Initial-boundary value problems for second-order parabolic equations
35K40	Second-order parabolic systems
35K50	Systems of parabolic equations, boundary value problems (MSC2000)
35K60	Nonlinear initial, boundary and initial-boundary value problems for linear parabolic equations

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degenerate parabolic equation; parabolic p-Laplace equation; singular parabolic equations; Hölder gradient estimate