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Characteristic classes of loop group bundles and generalized string classes. (English)

Zbl 0806.57014

Szenthe, J. (ed.) et al., Differential geometry and its applications. Proceedings of a colloquium, held in Eger, Hungary, August 20-25, 1989, organized by the János Bolyai Mathematical Society. Amsterdam: North- Holland Publishing Company. Colloq. Math. Soc. János Bolyai. 56, 33-66 (1992).

Let M be a smooth manifold, G a Lie group with the Lie algebra $\mathfrak{g}$, $\xi = \{g_{UV}\}$ a G-bundle over M. Then $\xi^{\#} = \{g^{\#}_{UV}\}$, $(g^{\#}_{UV}(\gamma))(t) = g_{UV}(\gamma(t))$, $\gamma \in \Omega M$, is an ΩG-bundle over ΩM. Here ΩM and ΩG are loop space and loop group over M and G, respectively. If ξ is the tangent bundle τ of M, then $\tau^{\#}$ is the tangent bundle of ΩM. So the study of loop group bundles has meanings. Let $\widetilde{\Omega}G$ be the basic central extension of ΩG. Then the obstruction $\widetilde{c}^1(\eta) \in H^2(X, S^1) \cong H^3(X, \mathbb{Z})$ for the lifting of the structure group of an ΩG-bundle $\eta = \{h_{UV}\}$, over a space X, to $\widetilde{\Omega}G$ has been defined and is named 'string class'. If X is ΩM and η is $\tau^{\#}$, then the transgression image of this class is the first Pontryagin class of M [*T. P. Killingback*, World-sheet anomalies and loop geometry, Nucl. Phys., B – Part. Phys. 288, 578-588 (1987)]. Precisely saying, the torsion part needs more delicate discussions [cf. *K. Pilch* and *N. P. Warner*, Commun. Math. Phys. 115, 191-212 (1988; Zbl 0661.58038)].

In this paper, assuming $G = U(n)$, characteristic classes $\widetilde{c}^p(\eta) \in H^{2p+1}(X, \mathbb{R})$, $p = 1, 2, \ldots$, of η are defined as the generalization of the (real) string class. Their definitions are as follows: Let $\{\theta_U\}$ be a connection form of η with the curvature form $\{\Theta_U\}$. Then there is a 0-cochain of $2p$-forms $\{\Psi_{p,U}\}$ such that $\int_0^1 \mathrm{tr}(\Theta^p_U g'_{UV} g^{-1}_{UV})dt = \Psi_{p,V} - \Psi_{p,U}$ for any p. Here $'$ means derivation with respect to the loop variable t. Then $\Phi_{p,U} = \int_0^1 \mathrm{tr}(\Theta^p_U \wedge \Theta'_U)dt - d\Psi_{p,U}$ defines a global closed $(2p+1)$-form on X and its de Rham class is determined by η. $\widetilde{c}^p(\eta)$ is defined to be the de Rham class of $\{\Phi_{p,U}\}$ (Lemma 6 and 7). It is shown that $\widetilde{c}^1(\eta)$ coincides with the above string class (as a real class, Theorem 3) and the following is shown:

(i) Define a G-bundle $\eta^\natural$ over $X \times S^1$ by $\eta^\natural = \{h^\natural_{UV}\}$, $h^\natural_{UV}(x,t) = (h_{UV}(x))(t)$; then $\widetilde{c}^p(\eta) = -(2\pi\sqrt{-1})^{p+1} p! \int_S 1\mathrm{Ch}^{p+1}(\eta^\natural)$d holds. Here $\mathrm{Ch}^{p+1}(\eta^\natural)$ is the $(p+1)$-th Chern character of $\eta^\natural$ (Theorem 2).

(ii) Let $\tau^{-1} : H^{q+1}(M, \mathbb{R}) \to H^q(\Omega M, \mathbb{R})$ be the inverse of the transgression map. Then $\widetilde{c}^p(\xi^{\#}) = -(2\pi\sqrt{-1})^{p+1} p! \tau^{-1}(\mathrm{Ch}^{p+1}(\xi))$ holds (Theorem 4).

(iii) η has the characteristic map $g : X \to G$ (Theorem 1) and

$$\widetilde{c}^p(\eta) = -(2\pi\sqrt{-1})^{p+1}(2p+1)!/(p+1)! g^*(e_{p+1})$$

holds. Here e_{p+1} is the $(p+1)$th generator of $H^*(U(n), \mathbb{Z})$ (Theorem 6).

To extend (iii) to the relation between string classes and Chern-Simons classes, the notion of 2-dimensional non-abelian de Rham cocycle with respect to ΩG is introduced. Then the above results are extended to 2-dimensional non-abelian de Rham cocycles with respect to ΩG. In this case, string classes may become fractional classes and the characteristic map may become a many-valued map. Detailed definitions of this cocycle and its connection and curvature are given in Section 1. Precise properties of $\widetilde{\Omega}\mathfrak{g}$-valued connections are also given. Here $\widetilde{\Omega}\mathfrak{g}$ means the basic central extension of $\Omega\mathfrak{g}$. Properties of the central extension part of a $\widetilde{\Omega}\mathfrak{g}$-valued connection give the prototype of the definition of $\widetilde{c}^p(\eta)$. The rest of the paper is devoted to the proofs of (i), (ii) and (iii).

For the entire collection see [Zbl 0764.00002].

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MSC:

57R20	Characteristic classes and numbers in differential topology	Cited in **4** Reviews
22E65	Infinite-dimensional Lie groups and their Lie algebras: general properties	
55R40	Homology of classifying spaces and characteristic classes in algebraic topology	
53C99	Global differential geometry	

Keywords:

associated loop group bundle; generalization of the string class; G- bundle; loop space; loop group; Pontryagin class; connection; Chern character; Chern-Simons classes; non-abelian de Rham cocycle