

**Borwein, David; Borwein, Jonathan M.; Girgensohn, Roland**

**Explicit evaluation of Euler sums.** (English) [Zbl 0819.40003](#)

Proc. Edinb. Math. Soc., II. Ser. 38, No. 2, 277-294 (1995).

In response to a letter from Goldbach, Euler considered sums of the form

$$\sigma_h(s, t) := \sum_{n=1}^{\infty} \left( 1 + \frac{1}{2^s} + \cdots + \frac{1}{(n-1)^s} \right) n^{-t},$$

where  $s$  and  $t$  are positive integers.

As Euler discovered by a process of extrapolation (from  $s + t \leq 13$ ),  $\sigma_h(s, t)$  can be evaluated in terms of Riemann  $\zeta$ - functions when  $s + t$  is odd. We provide a rigorous proof of Euler's discovery and then give analogous evaluations with proofs for corresponding alternating sums. Relatedly we give a formula for the series

$$\sum_{n=1}^{\infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} \right)^2 (n+1)^{-m}.$$

This evaluation involves  $\zeta$ - functions and  $\sigma_h(2, m)$ .

Reviewer: D.Borwein (London/Ontario), J.M.Borwein (Burnaby), R.Girgensohn (Waterloo)

#### MSC:

[40A25](#) Approximation to limiting values (summation of series, etc.)

[40B05](#) Multiple sequences and series

[11M99](#) Zeta and  $L$ -functions: analytic theory

[33E99](#) Other special functions

Cited in **6** Reviews  
Cited in **94** Documents

#### Keywords:

Riemann zeta function; beta function; psi function; generating functions; polylogarithms; harmonic numbers

**Full Text:** [DOI](#)

#### References:

- [1] DOI: 10.1016/0377-0427(91)90112-W · [Zbl 0782.33001](#) · doi:10.1016/0377-0427(91)90112-W
- [2] Berndt, Ramanujan's Notebooks, Part I (1985) · [Zbl 0555.10001](#) · doi:10.1007/978-1-4612-1088-7
- [3] Bailey, Experiment. Math. 3 pp 17– (1994) · [Zbl 0810.11076](#) · doi:10.1080/10586458.1994.10504573
- [4] Stromberg, An Introduction to Classical Real Analysis (1981) · [Zbl 0454.26001](#)
- [5] Euler, Opera Omnia, Ser 1 XV pp 217– (1917)
- [6] DOI: 10.1016/0022-314X(87)90012-6 · [Zbl 0606.10032](#) · doi:10.1016/0022-314X(87)90012-6
- [7] Nielsen, Die Gammafunktion (1965)
- [8] Lewin, Polylogarithms and Associated Functions (1981)
- [9] Hoffman, Pacific J. Math. 152 pp 275– (1992) · [Zbl 0763.11037](#) · doi:10.2140/pjm.1992.152.275
- [10] Rao, Pacific J. Math. 113 pp 471– (1984) · [Zbl 0549.10031](#) · doi:10.2140/pjm.1984.113.471

This reference list is based on information provided by the publisher or from digital mathematics libraries. Its items are heuristically matched to zbMATH identifiers and may contain data conversion errors. It attempts to reflect the references listed in the original paper as accurately as possible without claiming the completeness or perfect precision of the matching.