Nualart, David
The Malliavin calculus and related topics. (English) Zbl 0837.60050
Probability and Its Applications. New York, NY: Springer-Verlag. xi, 266 p. (1995).

This book is a very readable introduction to the Malliavin calculus and some of its applications. Its origins are series of lectures by the author on several occasions and this is pleasantly felt throughout in the excellent pedagogic exposition. The book presupposes some familiarity with martingale theory and Itô stochastic calculus, but readers with a good general background in probability will also be able to follow the exposition. An outline of the contents is as follows.

Chapter 1 is an introduction to "Analysis on the Wiener space", presented in the framework of a general probability space (as opposed to the formulation with an abstract Wiener space). It starts with the Wiener chaos decomposition, multiple stochastic integrals and Itô stochastic integrals. The derivative operator D and the Skorokhod integral δ as its adjoint are defined and their properties are studied. In this part, the formulation in terms of chaos expansions makes the proofs in general very transparent. Afterwards, the Ornstein-Uhlenbeck semigroup and its generator L are introduced and studied in detail. This includes the hypercontractivity property, the multiplier theorem and the relations between L, D and δ. At the end of the chapter, Meyer's inequalities are proved by the method of Pisier, relying on the boundedness of the Hilbert transform. A well-known consequence is the continuity of δ and L.

Chapter 2 discusses the "Smoothness of probability laws". A first section presents two sets of criteria for existence and/or smoothness of densities for random variables. The first is based on the integration by parts formula for the derivative operator D and allows results on both existence and smoothness of densities. The second uses the approach of Bouleau and Hirsch to establish existence of a density under some weaker assumptions. One major application of the Malliavin calculus is a probabilistic proof of Hörmander's theorem that under a nondegeneracy assumption on its coefficients, the solution of a stochastic differential equation has an absolutely continuous law; the density is smooth if the coefficients are sufficiently regular. This result is proved here following the method of Norris and forms the main part of Chapter 2. Another application is the existence of a density for the solution of a stochastic partial differential equation; this is proved here for a hyperbolic case by an argument similar to the proof of Hörmander's theorem and for the stochastic heat equation with the method of Bouleau and Hirsch.

Chapter 3 is devoted to an "Anticipating stochastic calculus". Stochastic integrals of non-adapted processes are defined as suitable limits of Riemann sums; this leads to both Skorokhod and Stratonovich stochastic integrals. The indefinite Skorokhod integral is then studied in more detail: continuity criteria, a result on quadratic variation and an Itô formula (whose proof is surprisingly long!) are established. Another idea to define anticipating stochastic integrals is based on the substitution method; this is explained and subsequently used to study Stratonovich stochastic differential equations with anticipating initial conditions or with boundary conditions.

Finally, Chapter 4 discusses "Transformations of the Wiener measure". The first half is devoted to anticipating Girsanov theorems, i.e., results on absolute continuity and the structure of the density of the measure obtained from Wiener measure by adding an anticipating drift. The main result is a theorem of Kusuoka which provides an explicit expression for the density under differentiability and invertibility assumptions on the drift. The second half of the chapter gives applications of this result to the study of the Markov random field property for solutions of stochastic differential equations with boundary conditions and of the germ Markov field property for solutions to stochastic partial differential equations. An alternative approach to the latter question is provided by a general characterization of conditional independence. This part of the book is rather technical and mainly addressed to specialists.

Each chapter contains some (nontrivial) exercises as well as comments on history and bibliography, mentioning in particular also topics not contained in this book. Among them are Watanabe's theory of distributions on the Wiener space, the more recent development of quasi-sure analysis and infinite-dimensional extensions. Further applications which are not discussed include small time asymptotics for heat kernels, filtering problems and processes with jumps. Nevertheless, the choice of topics in the book is such that it conveys an impression of the power of the Malliavin calculus while staying within reasonable bounds regarding the number of pages. I enjoyed reading the book and I can recommend it to anyone who is interested in its subject.

Reviewer: M.Schweizer (Berlin)

MSC:

60H07	Stochastic calculus of variations and the Malliavin calculus
60-02	Research exposition (monographs, survey articles) pertaining to probability theory
60H05	Stochastic integrals
60H10	Stochastic ordinary differential equations (aspects of stochastic analysis)
60H15	Stochastic partial differential equations (aspects of stochastic analysis)
60J25	Continuous-time Markov processes on general state spaces

Keywords:

Wiener space; Skorokhod integral; Meyer's inequalities; Hilbert transform; smoothness; Malliavin calculus; Hörmander's theorem; stochastic partial differential equation; heat equation; anticipating stochastic calculus; stochastic integrals; Itô formula; Wiener measure; Girsanov theorems; Markov random field