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Cauchy singular integrals and rectifiability of measures in the plane. (English) Zbl 0842.30029

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The author studies the relation between the rectifiability properties of measures on the complex plane $\mathbb{C}$ and their singular Cauchy transforms. Let μ be a finite non-negative Borel measure on $\mathbb{C}$, with spt μ as its support, and consider its Cauchy transform C_μ, given by

$$C_\mu(z) = \int (\zeta - z)^{-1} d\mu(\zeta), \qquad z \in \mathbb{C}.$$

On $\mathbb{C} \backslash \operatorname{spt} \mu$, C_μ represents a holomorphic function, while for $z \in \operatorname{spt} \mu$, $C_\mu(z)$ may exist in the principal value sense. More precisely, for $a \in \mathbb{C}$ and $r > 0$, we set

$$B(a:r) = \{z \in \mathbb{C} : |z-a| \leq r\},\ E(a:r) = \{z \in \mathbb{C} : |z-a| \geq r\},$$

and we define

$$C_\mu(a) = \lim_{r \to 0^+} \int_{E(a:r)} (z-a)^{-1} d\mu(z),$$

provided the limit exists and is finite. The main result of the paper is that if for μ-almost all $a \in \mathbb{C}$, $C_\mu(a)$ exists and is finite, and

$$\varliminf_{r \to 0^+} \frac{\mu(B(a:r))}{r} > 0,$$

then μ is concentrated on countably many rectifiable curves, i.e. μ is rectifiable in the sense that there is a sequence (Γ_k) of rectifiable curves so that

$$\mu\left(\mathbb{C} \backslash \bigcup_{k=1}^{\infty} \Gamma_k\right) = 0.$$

A natural question that remains open is whether the lower limit in the above assumption could be replaced by the upper limit. The method of proof is based on the concept of tangent measures introduced by *D. Preiss* [Ann. Math., II. Ser. 125, 537-643 (1987; Zbl 0627.28008)].

After the completion of the present paper, the author, together with D. Preiss, has extended the present result to the higher-dimensional setting of $\mathbb{R}^n$. These higher-dimensional results are reported in the paper "Rectifiable measures in $\mathbb{R}^n$ and existence of principal values for singular integrals" which will appear in the J. Lond. Math. Soc.

Reviewer: J.Burbea (Pittsburgh)

MSC:

30E20 Integration, integrals of Cauchy type, integral representations of analytic functions in the complex plane

28A15 Abstract differentiation theory, differentiation of set functions

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Cauchy singular integral; rectifiable measures; principal value; tangent measure; Hausdorff measure

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