Hovey, Mark
Model categories. (English) Zbl 0909.55001
Mathematical Surveys and Monographs. 63. Providence, RI: American Mathematical Society (AMS). xii, 209 p. (1999).

In recent years, *D. G. Quillen*'s homotopical algebra [Homotopical algebra, Lect Notes Math. 43 (1967; Zbl 0168.20903)] has gained increasing popularity. The book under review gives a modern and accessible account of the basic facts; and even if it is not intended to be a textbook, it should be a good starting point for students, as well as a reference for active researchers. The book fills a vacant niche in the literature, and although it has some unpublished competitors, it may well become a standard reference. In this respect its short bibliography is a weakness. There are a number of slight variations of the standard definitions (even the definition of the title, "model categories" is somewhat different from the original), but the author is careful to point them out, and this should not be a problem for most readers.

The problem addressed is the following: suppose one has a category where certain maps, the "weak equivalences", give a very close connection between the source and target. The weak equivalences need not be isomorphisms, but we would like to consider them as such. Does it make sense to just formally invert the weak equivalences? Quillen's homotopical algebra, or the theory of closed model categories (which has become fashionable to abbreviate to "model categories") gives a framework within which this question can be answered in the affirmative, and where we have some control over the category even after inverting the weak equivalences.

A model category is a category $\mathcal{C}$ with three distinguished subcategories, whose maps are called respectively, fibrations, cofibrations and weak equivalences. These maps are required to satisfy the usual lifting properties of homotopy theory: if

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & X \end{array}$$

is a diagram in $\mathcal{C}$ such that i is a cofibration and f is a fibration, and if either i or f is also a weak equivalence, then there is a lifting $B@>>>Y$ making the resulting diagram commutative. There are additional requirements, with slight modifications according to which source you consult. Given this, one can form the homotopy category Ho$\mathcal{C}$. For some applications it is enough to consider only the homotopy category, but for others it is vital that one keeps all the structure provided by the model category. The latter accounts for much of the popularity of the subject.

Examples of model categories are the categories of topological spaces, simplicial sets, chain complexes, any of the popular versions of the category of spectra, Morel and Voevodsky's A^1-homotopy theory and so on.

The plan of the book is as follows: The first chapter is devoted to setting the stage, and gives the basic definitions. In the second chapter some examples are listed, and in the third the perhaps most fundamental example, the category of simplicial sets, is treated to some length. In the fourth chapter, the author discusses how a monoidal structure and a model structure may coexist in a category, giving rise to a monoidal model category, and modules over monoidal model categories. The prime example of a monoidal model category is again simplicial sets, and modules over the category of simplicial sets are called simplicial model categories (this is slightly different from Quillen's original definition of closed simplicial model categories). In the fifth chapter, the author shows that simplicial sets sit at the core of the theory: the homotopy category of a model category is a module over the homotopy category of simplicial sets. This comes from the ability to form simplicial and cosimplicial resolutions, a point developed by *W. G. Dwyer* and *D. M. Kan* in [Topology 19, No. 4, 427-440 (1980; Zbl 0438.55011)]. In the sixth chapter, pointed structures are discussed, and it is shown that in this case the homotopy category is a module over the homotopy category of pointed simplicial sets. Adding some crucial conditions, the author ends up with what he calls a pre-triangulated category. If the suspension functor is an equivalence, he gets something which he calls a triangulated category, which is not quite the usual notion. This situation is considered in the seventh chapter. This closes in on the original goal of the author, which was to determine when the homotopy category of a model category is a stable homotopy category in the sense of *M. Hovey, J. H. Palmieri* and *N. P. Strickland* [Axiomatic stable homotopy theory, Mem. Am. Math. Soc. 610 (1997; Zbl 0881.55001)]. The book ends with the eighth chapter, in which the author invites the reader to ponder upon some unsolved questions.

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MSC:

55-02	Research exposition (monographs, survey articles) pertaining to algebraic topology
18G30	Simplicial sets; simplicial objects in a category (MSC2010)
55U35	Abstract and axiomatic homotopy theory in algebraic topology

Keywords:

homotopical algebra; model categories; stable categories; homotopy theory; category of simplicial sets