

**Buchstaber, V. M.; Enolskii, V. Z.; Leykin, D. V.**

**Kleinian functions, hyperelliptic Jacobians and applications.** (English) Zbl 0911.14019

Rev. Math. Math. Phys. 10, No. 2, 3-120 (1997).

The treatise under review provides a fairly comprehensive account of the classical theory of hyperelliptic Kleinian functions, together with some new applications towards the description of hyperelliptic Jacobians and their Kummer varieties as well as to families of matrix differential operators and their associated soliton equations. The text consists of six chapters, one appendix, and a rich bibliography referring to the classical and to the contemporary literature on abelian functions and their use in solving various classes of nonlinear differential equations of KdV-type.

Chapter I presents a constructive approach to F. Klein's hyperelliptic sigma functions [cf. *F. Klein*, Math. Ann. 32, 351–380 (1888; [JFM 20.0491.01](#))], which is based on the modern concept of the moduli space of  $n$ -dimensional principally polarized abelian varieties and the universal bundle over it. In this framework, Klein's classical sigma functions appear as automorphic functions on the moduli subspace of Jacobians of plane hyperelliptic curves, and their properties are discussed, in greater detail, in chapter II. Chapter III compiles some background material for the theory of Kleinian functions, most of which is thoroughly treated in *H. F. Baker's* brilliant classic "Abel's theorem and the allied theory of theta functions" (Cambridge Univ. Press 1897; reprint 1995; [Zbl 0848.14012](#)).

Chapter IV is devoted to the basic relations between the various Kleinian functions and their derivatives. These relations are then applied to the modern theory of integrable Hamiltonian systems, in particular to the construction of explicit solutions of the Korteweg–de Vries system (KdV) in terms of Kleinian functions on moduli spaces of hyperelliptic Jacobians, and to the problem of constructing families of matrix differential operators satisfying certain curvature conditions. Chapter V gives a more subtle analysis of the fundamental cubic and quartic relations for Kleinian functions derived in chapter IV. These deeper results are shown to lead to explicit matrix realizations of hyperelliptic Jacobians and their associated Kummer varieties, on the one hand, and to the description of a certain dynamical system defined on the universal space of Jacobians of hyperelliptic curves of given genus  $g$ . Also, this construction allows to establish systems of linear differential operators for which the hyperelliptic base curve is their common spectral variety. – In chapter VI, the authors study abelian Bloch functions, as solutions of Schrödinger equations, which are expressible in terms of Kleinian functions of genus 2. This investigation culminates in a new addition theorem for the Akhiezer-Baker function on the Jacobian of a genus-2 curve. In addition, spectral problems for singular Kummer surfaces and Humbert surfaces are discussed in this context.

The appendix to the paper gives a brief survey on some very recently discovered addition theorems for hyperelliptic Kleinian functions, which are due to the authors and will be discussed, in greater detail, in a forthcoming paper. Altogether, the comprehensive article under review is based on recent results of the authors, which were partially announced in several short notes, contributions to conference proceedings, and preprints published during the past four years, and which are presented here in a thorough, detailed and coherent fashion. The paper contains a wealth of classical and new aspects of the theory of abelian functions and their applications to differential equations of mathematical physics.

Reviewer: [Werner Kleinert \(Berlin\)](#)

#### MSC:

- [14K20](#) Analytic theory of abelian varieties; abelian integrals and differentials
- [14H42](#) Theta functions and curves; Schottky problem
- [35Q53](#) KdV equations (Korteweg-de Vries equations)
- [14H40](#) Jacobians, Prym varieties
- [14K25](#) Theta functions and abelian varieties
- [14G10](#) Zeta functions and related questions in algebraic geometry (e.g., Birch-Swinnerton-Dyer conjecture)

Cited in <b>2</b> Reviews Cited in <b>45</b> Documents
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#### Keywords:

[hyperelliptic Kleinian functions](#); [Kummer varieties](#); [soliton equations](#); [Klein's hyperelliptic sigma func-](#)

tions; JFM 20.0491.01; matrix realizations of hyperelliptic Jacobians