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Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. (English)

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Mathematics in Science and Engineering. 198. San Diego, CA: Academic Press. xxiv, 340 p. (1999).

The subject of fractional calculus (that is, derivatives and integrals of any real or complex order) has gained importance during the past three decades due mainly to its demonstrated applications in such diverse fields of science and engineering as (for example) fluid flow, rheology, dynamical processes in self-similar and porous structures, diffusive transport akin to diffusion, electrical networks, probability and statistics, control theory of dynamical systems, viscoelasticity, electrochemistry of corrosion, and chemical physics. Indeed it provides several potentially useful tools for solving differential and integral equations, and other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables.

The concept of fractional calculus seems to have stemmed from a letter from Marquis de l'Hôpital (1661-1704) to G. W. Leibniz (1646-1716), dated 30 September 1695, which sought the meaning of Leibniz's (currently popular) notation $\frac{d^n y}{dx^n}$ for the derivative of order n when $n = \frac{1}{2}$. Fractional integrals of an arbitrary (real or complex) order are a generalization of the ordinary integral of order $n (n \in \mathbb{N})$. Indeed, if we define the linear integral operators $\mathcal{I}$ and $\mathcal{K}$ by

$$(\mathcal{I}f)(x) := \int_0^x f(t)dt \tag{1}$$

and

$$(\mathcal{K}f)(x) := \int_x^\infty f(t)dt, \tag{2}$$

then it is easily seen by iteration that

$$(\mathcal{I}^n f)(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)dt \quad (n \in \mathbb{N}) \tag{3}$$

and

$$(\mathcal{K}^n f)(x) = \frac{1}{(n-1)!}\int_x^\infty (t-x)^{n-1} f(t)dt \quad (n \in \mathbb{N}). \tag{4}$$

With a view to interpolating $(n-1)!$ between the positive integer values of n, one can set

$$(n-1)! = \Gamma(n) \tag{5}$$

in terms of the familiar Gamma function. Thus, in general, equations (3) and (4) would lead eventually to the Riemann-Liouville operator $\mathcal{R}^\mu$ and the Weyl operator $\mathcal{W}^\mu$ of fractional integral of order μ $(\mu \in \mathbb{C})$, defined by

$$(\mathcal{R}^\mu f)f(x) := \frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1} f(t)dt \quad \big(\Re(\mu) > 0\big) \tag{6}$$

and

$$(\mathcal{W}^\mu f)(x) := \frac{1}{\Gamma(\mu)}\int_x^\infty (t-x)^{\mu-1} f(t)dt \quad \big(\Re(\mu) > 0\big), \tag{7}$$

respectively, it being assumed that the function $f(t)$ is so constrained that the integrals in (6) and (7) exist. And there are operators of fractional derivatives $\mathcal{D}^\mu_{x;0}$ and $\mathcal{D}^\mu_{x;\infty}$ of order μ $(\mu \in \mathbb{C})$, which correspond to the fractional integral operators $\mathcal{R}^\mu$ and $\mathcal{W}^\mu$, respectively, and we have

$$(\mathcal{D}^\mu_{x;0} f)(x) := \frac{d^m}{dx^m}(\mathcal{R}^{m-\mu} f)(x) \quad (0 \le \Re(\mu) < m;\ m \in \mathbb{N}) \tag{8}$$

and

$$(\mathcal{D}^{\mu}_{x;\infty} f)(x) := \frac{d^m}{dx^m} (\mathcal{W}^{m-\mu} f)(x) \quad (0 \leq \Re(\mu) < m;\ m \in \mathbb{N}). \tag{9}$$

There also exist, in the literature on fractional calculus, numerous further extensions and generalizations of the operators $\mathcal{R}^{\mu}$, $\mathcal{W}^{\mu}$, $\mathcal{D}^{\mu}_{x;0}$, and $\mathcal{D}^{\mu}_{x;\infty}$, each of which we have chosen to introduce here for the sake of nonspecialists in this subject.

The book is written by an applied mathematician for potential users of fractional calculus. The subject-matter of this book is divided into ten chapters. Chapter 1 provides some basic theory of the various special functions which are used in the latter chapters. The author treats, in this chapter, the Gamma and Beta functions, the Mittag-Leffler function and its generalizations, and Wright's familiar extension of the classical Bessel functions $J_\nu(z)$ and $I_\nu(z)$. An additional section on the H-function of Charles Fox (1897-1977) would have provided a unified approach to the Mittag-Leffler and Wright functions.

Chapter 2 introduces the concept of fractional calculus in a lucid manner, and presents various properties and characteristics of fractional derivatives (especially of the Riemann-Liouville fractional derivative).

In Chapters 3 to 8, the author concentrates upon the theory and the methods of solution of fractional differential equations. In particular, existence and uniqueness theorems are discussed in Chapter 3, the Laplace transform and the Green-function methods of solutions are considered in Chapters 4 and 5, respectively, other methods based (for example) on Mellin transforms, symbolic calculus, and orthogonal polynomials are summarized in Chapter 6, and computational aspects of fractional derivatives and numerical methods for solving fractional differential equations are presented in Chapters 7 and 8, respectively. Finally, in Chapters 9 and 10, the author gives a survey of a wide variety of applications of fractional calculus, including some of those that are mentioned above.

This book contains an appendix providing useful tables of fractional derivatives, a bibliography citing as many as 259 references (most [if not all] of which are actually referred to in the text of the book), and a four-page index at the end of the book.

This is by no means the first (or the last) book on the subject of fractional calculus, but indeed it is one that would undoubtedly attract the attention (and successfully serve the needs) of mathematical, physical, and engineering scientists looking for applications of fractional calculus. I, therefore, recommend this well-written book to all users of fractional calculus.

Reviewer: H.M.Srivastava (Victoria, British Columbia)

MSC:

34A25	Analytical theory of ordinary differential equations: series, transformations, transforms, operational calculus, etc.
26A33	Fractional derivatives and integrals
34Axx	General theory for ordinary differential equations
34-01	Introductory exposition (textbooks, tutorial papers, etc.) pertaining to ordinary differential equations

Keywords:

fractional differential equations; fractional derivatives; existence; fractional calculus; Riemann-Liouville operator; Weyl operator; fractional integral; uniqueness