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Cohomology of finite group schemes over a field. (English) Zbl 0945.14028
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From the introduction: A finite group scheme G over a field k is equivalent to its coordinate algebra, a finite dimensional commutative Hopf algebra $k[G]$ over k. In many contexts, it is natural to consider the rational (or Hochschild) cohomology of G with coefficients in a $k[G]$-comodule M. This is naturally isomorphic to the cohomology of the dual cocommutative Hopf algebra $k[G]^{\#}$ with coefficients in the $k[G]^{\#}$-module M. In this latter formulation, we encounter familiar examples of the cohomology of group algebras $k\pi$ of finite groups π and of restricted enveloping algebras $V(g)$ of finite dimensional restricted Lie algebras g. – In recent years, the representation theory of the algebras $k\pi$ and $V(g)$ has been studied by considering the spectrum of the cohomology algebra with coefficients in the ground field k and the support in this spectrum of the cohomology with coefficients in various modules. This approach relies on the fact that $H^*(\pi, k)$ and $H^*(V(g), k)$ are finitely generated k-algebras. Rational representations of algebraic groups in positive characteristic correspond to representations of a hierarchy of finite group schemes. In order to begin the process of introducing geometric methods to the study of these other group schemes, finite generation must be proved. Such a proof has proved surprisingly elusive [though partial results can be found in a paper by *E. M. Friedlander* and *B. J. Parshall*, Math. Ann. 273, 353-374 (1986; Zbl 0586.20021)]. – The main theorem of this paper is the following:

Theorem 1.1. Let G be a finite group scheme and M a finite dimensional rational G-module. Then $H^*(G, k)$ is a finitely generated k-algebra and $H^*(G, M)$ is a finite $H^*(G, k)$-module.

Our proof of finite generation is quite constructive. We embed G in some general linear group GL_n and establish the existence of universal extension classes for GL_n of specified degrees. In a direct manner, these classes provide the generators of $H^*(G, k)$. – In order to construct these universal extension classes, we follow closely the approach of *V. Franjou*, *J. Lannes*, and *L. Schwartz* [Invent. Math. 115, No. 3, 513-538 (1994; Zbl 0798.18009)].

We briefly summarize the contents of this paper. In section 1, we show that theorem 1.1 is implied by the existence of certain universal extension classes for general linear groups over fields of positive characteristic p. The universality of these classes suggest that they should have independent interest. Our category $\mathcal{P}$ of strict polynomial functors of finite degree is introduced in section 2 and its basic properties are proved. These include the existence of explicit projective and injective generators and a vanishing result crucial for our calculations of Ext-groups. Related foundational results were previously established by *N. J. Kuhn* [e.g. Am. J. Math. 116, No. 2, 327-360 (1994; Zbl 0813.20049)]. The relationship between strict polynomial functors of finite degree and modules over classical Schur algebras is made explicit in section 3 using the properties of $\mathcal{P}$ established in section 2. Following *S. Donkin* [J. Algebra 104, 310-328 (1986; Zbl 0606.20038)], we also give a proof of the close relation between cohomology of Schur algebras and rational cohomology of general linear groups. In section 4, we present a somewhat shortened version of the ingenious constructions of Franjou-Lannes-Schwartz (loc. cit.) which establishes the existence of our fundamental extensions. Necessary further computations of Ext-groups between polynomial functors are provided in section 5, enabling us to identify in section 6 the restrictions of our universal classes to certain infinitesimal sub-group schemes of GL_n. At this point, the proof of theorem 1.1 is complete.

We continue the paper with our computation of $H_*(GL(\mathbb{F}_q), M(\mathbb{F}_q))$ in section 7. Finally, in section 8 we construct (in the special case of fields of characteristic 2) explicit injective resolutions of functors $S^{m(r)}$ which give immediate computations of the corresponding Ext-groups and provide, in particular, representatives for our universal extensions.

MSC:

14L15	Group schemes	Cited in **17** Reviews
20G15	Linear algebraic groups over arbitrary fields	Cited in **114** Documents

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