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Filtrations on higher algebraic K-theory. (English) Zbl 0951.19003

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Let X be a noetherian scheme and let $K_0(X)$ denote its Grothendieck group of locally free coherent $\mathcal{O}_X$-modules. On $K_0(X)$ one has the filtration by codimension of supports F^p_{cod} and also the γ-filtration F^p_γ. It has been known since a long time that $F^p_\gamma K_0(X) \subset F^p_{\text{cod}} K_0(X)$. One also has these filtrations on the higher algebraic K-groups $K_m(X)$. Here a detailed account is given of the fact that one has, for X of finite type over a field, the inclusions $F^p_\gamma K_m(X) \subset F^{p-m}_{\text{cod}} K_m(X)$. To prove this result a third filtration, the so called Brown filtration F^p of $K_m(X)$, is introduced and both F^p_γ and F^p_{cod} are related to F^p. The paper consists of an introduction, seven sections and a bibliography.

The first section deals with the general setup of homotopy theory of simplicial sheaves on a Grothendieck site C. It is known that the category $s\mathcal{T}$ of sheaves of simplicial sets on C is a closed simplicial model category. Its objects are called spaces. One may consider the subcategory $s\mathcal{G} \subset s\mathcal{T}$ of simplicial sheaves of groups and group homomorphisms. It is shown in detail that $s\mathcal{G}$ is a closed model category as well. So one has a good homotopy theory for $s\mathcal{G}$. One has the notion of Postnikov tower, Eilenberg-MacLane spaces $K(\pi, n)$, cohomology $H^n(X, \pi)$, etc. For two spaces X and (pointed) Y one has the Brown cohomological spectral sequence $E^{p,q}_r$, $r \geq 2$, given by $E^{p,q}_2 = H^p(X, \pi_{-q}Y)$ for $p + q \leq 0$, and $E^{p,q}_2 = 0$ else. The E_∞ is related to $H^{p+q}(X, Y)$.

The second section gives the comparison of two spectral sequences. Spectral sequences are presented by means of exact couples and it is shown how under a shift of filtration an isomorphism on E_2 terms is induced. Returning to geometry, let X be a finite-dimensional noetherian topological space with a sheaf G of simplicial groups on X. One may compute the homotopy type of the simplicial group $\Gamma(X, G) = \operatorname{Hom}(X, G)$ by means of the Postnikov tower of G. One may also compute $\Gamma(X, G)$ via the filtration by coniveau (or codimension). The first point of view leads to the corresponding Brown spectral sequence $E^{p,q}_r = E^{p,q}_r(X, G)$ while the second gives the sequence $E^{p,q}_{r,\text{cod}} = E^{p,q}_{r,\text{cod}}(X, G)$. Under suitable conditions on G and Γ the main result of this section gives a morphism of spectral sequences $E^{p,q}_r \to E^{p,q}_{r,\text{cod}}$ compatible with the identity map on the abutments $H^{-p-q}(X, G)$. In particular, one has inclusions $F^p H^m(X, G) \subset F^p_{\text{cod}} H^m(X, G)$.

The next step is to implicate higher algebraic K-theory. K-theory for a locally ringed topos is defined. The notion of K-coherence for a space X is introduced. Let T be the category of sheaves on a category of schemes $\mathcal{C}$. For $X \in \mathcal{C}$ let $P(X)$ denote the category of locally free and finitely generated $\mathcal{O}_{T/X}$-modules and let ΩBQP be the simplicial sheaf associated to that functor. Using a fibrant resolution of ΩBQP one has a natural map $K_m(X) = \pi_m(\Omega BQP(X)) \to H^{-m}(X, K)$, $K = \mathbb{Z} \times \mathbb{Z}_\infty \mathrm{BGL}$, for all $m \geq 0$. For X a noetherian scheme of finite Krull dimension, and T either the big Zariski topos of all noetherian schemes of finite Krull dimension, or the big Zariski topos of all schemes of finite type over X or the small Zariski topos of X, it is shown that $K_m(X) \to H^{-m}(X, K)$ is an isomorphism. For a field k, let $\mathcal{V}$ denote the category of regular schemes of finite type over $S = \operatorname{Spec}(k)$ equipped with the Zariski topology. Let $s\mathcal{V}$ be the category of simplicial objects in $\mathcal{V}$. For $X \in s\mathcal{V}$, let E be a (finite rank) vector bundle on X with associated projective bundle $\pi : P(E) \to X$. Then, for the filtration F^r associated to the Brown spectral sequence, one has $(\pi^*)^{-1}(F^r K_m(P(E))) = F^r K_m(X)$.

In the fourth section it is shown that for a K-coherent space X the $H^0(X, K)$ is a λ-algebra with involution and augmentation $\varepsilon : H^0(X, K) \to H^0(X, \mathbb{Z})$ induced by projection onto the first factor of $\mathbb{Z} \times \mathbb{Z}_\infty \mathrm{BGL}$. For $m > 0$, $H^{-m}(X, K)$ is an $H^0(X, K)$-λ-module with involution.

Section 5 deals with the Brown-Gersten spectral sequence and its relation to the Quillen spectral sequence. It leads to the main results of the paper: (i) For all $m \geq 0$ and $p \geq 0$ one has $F^p K_m(X) \subset F^p_{\text{cod}}(X)$; (ii) When X is of finite type over a field, then $F^p_\gamma K_m(X) \subset F^{p-m} K_m(X)$; (iii) When X is regular of finite type over a field the Quillen spectral sequence coincides from E_2 on with the Brown-Gersten spectral sequence. In particular, $F^p K_m(X) = F^p_{\text{cod}} K_m(X)$.

Section 6 relates Chern classes and the λ-ring structure on higher K-groups to the effect that for a K-coherent space the total Chern class is a morphism of λ-rings with involution.

In the final section the Bloch-Lichtenbaum spectral sequence for a field F (tensored with $\mathbb{Q}$) is shown to degenerate from E_2 on, and to converge to the γ-filtration on $K_{-p-q}(F) \otimes_{\mathbb{Z}} \mathbb{Q}$.

For the entire collection see [Zbl 0931.00031].

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