Alzer, Horst

A power mean inequality for the gamma function. (English) Zbl 0964.33002

Monatsh. Math. 131, No. 3, 179-188 (2000).

In this interesting paper, the author extends a result due to *L. G. Lucht* [Aequationes Math. 39, No. 2/3, 204-209 (1990; Zbl 0705.39002)] on convexity-like inequalities for Euler's gamma function, involving the geometric mean.

Let $x_j > 0,\ p_j > 0\ (1 \le j \le n),\ p_1 + \cdots + p_n = 1$ and $n \ge 2$. Then the old result can be stated as:

$$\Gamma\left(x_1^{p_1} \cdots x_n^{p_n}\right) \le \left(\Gamma\left(x_1^{p_1}\right)\right) \cdots \left(\Gamma\left(x_n^{p_n}\right)\right)$$

for all $x_j \ge a$ or $\le a$, with a the unique positive zero of $\psi(x) + x\psi'(x)$. Equality holds if and only if $x_1 = \cdots = x_n$.

In the paper the *power mean* for the numbers $\{x_j\}$, weights $\{p_j\}$ and a real parameter r is defined as

$$M_n^{[r]}(x_j, p_j) = \left(\sum_{j=1}^{n} p_j x_j^r\right)^{1/r} \qquad (r \ne 0), \qquad M_n^{[0]}(x_j, p_j) = \prod_{j=1}^{n} x_j^{p_j}.$$

As the gamma function is strictly convex for $x > 0$, we have the well known inequality

$$\Gamma(p_1x_1 + \cdots p_nx_n) \le p_1\Gamma(x_1) + \cdots p_n\Gamma(x_n),$$

which can now be seen as

$$\Gamma\left(M_n^{[1]}(x_j, p_j)\right) \le M_n^{[1]}(\Gamma(x_j), p_j).$$

The author now introduces the following real numbers: $r_0 = 0.21609\ldots$: the unique positive zero of $\psi(x) + x\psi'(x)$, $r_1 = 1.46163\ldots$: the unique positive zero of $\psi(x)$, $r_2 = 2.08907\ldots$: the unique positive zero of $x\psi(x) - 1$,

$$\begin{aligned}\alpha = 0.01317\ldots &= \sup_{0<x<r_0} \left\{\psi(x) + x\psi'(x)\right\} / \left\{\psi(x) - x(\psi(x))^2\right\},\\ \beta = 11.29416\ldots &= \inf_{r_1<x<r_2} \left\{\psi(x) + x\psi'(x)\right\} / \left\{\psi(x) - x(\psi(x))^2\right\}.\end{aligned}$$

The main result of the paper is:

$$\Gamma\left(M_n^{[r]}(x_j, p_j)\right) \le M_n^{[r]}(\Gamma(x_j), p_j)$$

holds for all positive n-tuples x_j and all positive weights p_j with sum 1, if and only if $\alpha \le r \le \beta$.

Furthermore some corollaries are given, one of which concerns the case $r = -1$: the classical harmonic mean. For this mean the author shows

$$\text{all } x_j \in [a, b] \Rightarrow \Gamma\left(M_n^{[-1]}(x_j, p_j)\right) \le M_n^{[-1]}(\Gamma(x_j), p_j),$$

$$\text{either all } x_j \in (0, a] \text{ or all } x_j \in [b, \infty) \Rightarrow \Gamma\left(M_n^{[-1]}(x_j, p_j)\right) \ge M_n^{[-1]}(\Gamma(x_j), p_j),$$

where $0 < a < b$ are the only positive zeros of $2\psi(x) - x(\psi(x))^2 + x\psi'(x)$. Again equality holds if and only if $x_1 = \cdots = x_n$.

Reviewer: Marcel G.de Bruin (Delft)

MSC:

33B15 Gamma, beta and polygamma functions
26D15 Inequalities for sums, series and integrals

Keywords:

gamma function; psi function; power means; complete monotonicity

Full Text: DOI