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l-independence of the trace of monodromy. (English) Zbl 0980.14014

Math. Ann. 315, No. 2, 321-340 (1999).

From the introduction: Let K be a complete discrete valuation field with finite residue field $\mathbb{F}_{p^h}$, G_K the absolute Galois group of K, I_K the inertia subgroup of G_K and W_K the Weil group of K. We define a subset W_K^+ of W_K to be $W_K^+ := \{g \in W_K | u(g) \geq 0\}$. Let X be a variety over K, $\overline{X}$ means the scalar extension $X \otimes_K \overline{K}$. We denote by l a prime number $\neq p$. We consider the traces of the action of elements of W_K^+ or W_K on the compact support étale cohomology $\mathrm{H}_c^i(\overline{X}, \mathbb{Q}_l) := \varprojlim_n \mathrm{H}_c^i(\overline{X}, \mathbb{Z}/l^n\mathbb{Z}) \bigotimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Let us recall the following classical conjecture [*J.-P. Serre* and *J. Tate*, Ann. Math. (2) 88, 492-517 (1968; Zbl 0172.46101)]:

For any variety X over K and $g \in W_K^+$, $\mathrm{Tr}(g^*; \mathrm{H}_c^i(\overline{X}, \mathbb{Q}_l))$ is a rational integer which is independent of the choice of l.

In this paper, we shall prove the following weak versions of the conjecture:

Proposition A. The trace $\mathrm{Tr}(g^*; \mathrm{H}_c^i(\overline{X}, \mathbb{Q}_l))$ is an algebraic integer for any variety X over K and any $g \in W_K^+$.

Theorem B. The alternating sum $\sum_{i=0}^{2d} (-1)^i \mathrm{Tr}(g^*; \mathrm{H}_c^i(\overline{X}, \mathbb{Q}_l))$ is a rational integer which is independent of l for any variety X over K and any $g \in W_K^+$.

When K is of mixed characteristics, we compare the traces for p-adic cohomologies and the traces for l-adic cohomologies in addition to the above result. Assume that X is proper and smooth over K. The étale cohomology $\mathrm{H}^i(\overline{X}, \mathbb{Q}_p) := \varprojlim_n \mathrm{H}^i(\overline{X}, \mathbb{Z}/p^n\mathbb{Z}) \bigotimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a potentially semi-stable representation by the works of A. J. de Jong and T. Tsuji. Fontaine attached a p-adic representation of the Weil(-Deligne) group $\widehat{D}_{\mathrm{pst}}(V)$ to a p-adic potentially semi-stable representation V. Thus we apply the functor $\widehat{D}_{\mathrm{pst}}$ on $\mathrm{H}^i(\overline{X}, \mathbb{Q}_p)$. Then $\widehat{D}_{\mathrm{pst}}(\mathrm{H}^i(\overline{X}, \mathbb{Q}_p))$ gives a "good" p-adic representation of the Weil group W_K which completes the family of l-adic cohomologies for $l \neq p$. We can prove:

Theorem D. For $g \in W_K^+$ and a proper smooth variety X over K, we have the following equality between rational integers:

$$\sum_i (-1)^i \mathrm{Tr}\big(g^*; \mathrm{H}^i(\overline{X}, \mathbb{Q}_l)\big) = \sum_i (-1)^i \mathrm{Tr}\Big(g^*; \widehat{D}_{\mathrm{pst}}\big(\mathrm{H}^i(\overline{X}, \mathbb{Q}_p)\big)\Big).$$

For the proof of theorem B, the result by A. J. de Jong concerning semi-stable reduction plays an essential role.

MSC:

14F20 Étale and other Grothendieck topologies and (co)homologies
14F30 p-adic cohomology, crystalline cohomology
14G15 Finite ground fields in algebraic geometry

Keywords:

trace of monodromy; geometric Frobenius; finite residue field; Weil group