Bownik, Marcin
The structure of shift-invariant subspaces of $L^2(\mathbb{R}^n)$. (English) Zbl 0986.46018
J. Funct. Anal. 177, No. 2, 282-309 (2000).

The author of this note investigates the structure of shift invariant spaces in $L^2(\mathbb{R}^n)$ under the action of some lattice $\Gamma = P\mathbb{Z}^n$, where P is a nonsingular n by n real matrix. He treates P as the unit matrix, since general P case follows by standard arguments.

The proofs are followed by the idea from *H. Helson's* book "Lectures on invariant subspaces", New York/London (1964; Zbl 0119.11303) and it also can reproduce the former results by *R. Ron* and *Z. Shen* of $L_2(\mathbb{R}^d)$, Can. J. Math. 47, No. 5, 1051-1094 (1995; Zbl 0838.42016).

The typical theorem is in the following:

Theorem. Suppose $V \subset L^2(\mathbb{R}^n)$ is shift invariant and J is its range function. For every shift preserving operator $L : V \to L^2(\mathbb{R}^n)$ there exists a measurable range operator R on J such that

$$(\mathcal{T} \circ L)f(x) = R(x)(\mathcal{T}f(x)) \quad \text{for a.e. } x \in \mathbb{T}^n, \quad f \in V, \qquad (*)$$

where $\mathcal{T} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n, \ell^2(\mathbb{Z}))$ defined for $f \in L^2(\mathbb{R}^n)$ by

$$\mathcal{T}f : \mathbb{T}^n \to \ell^2(\mathbb{Z}^n), \quad \mathcal{T}f(x) = (\widehat{f}(x+k))_{k\in\mathbb{Z}^n},$$

is an isometric isomorphic between $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n))$. Conversely, given a measurable range operator R on J with $\sup \mathrm{ess}_{x\in\mathbb{T}^n} \|R(x)\| < \infty$ there is a bounded shift preserving operator $L : V \to L^2(\mathbb{R}^n)$ such that $(*)$ holds. The correspondence between L and R is one-to-one under the convention that the range operators are identified if they are equal a.e. Moreover, we have $\|L\| = \sup \mathrm{ess}_{x\in\mathbb{T}^n} \|R(x)\|$.

Reviewer: Shozo Koshi (Sapporo)

MSC:

46E30 Spaces of measurable functions (L^p-spaces, Orlicz spaces, Köthe function spaces, Lorentz spaces, rearrangement invariant spaces, ideal spaces, etc.)

Keywords:
range function; shift-preserving operator; frame; Riesz family; shift invariant spaces

Full Text: DOI

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