Schröder, Karl-Heinz

Locally convex algebras. (English) Zbl 1010.46046

Note Mat. 19, No. 1, 87-151 (1999).

The paper under review (the author's thesis) consists of four Chapters (from 0 to 3). The notations are given in Chapter 0 and the main notions, illustrative examples and auxiliary results in Chapter 1. All topological algebras considered here have a jointly continuous multiplication. Examples of topological algebras A, which are not idempotent topological algebras, or whose topological dual space $A' = \{\theta_A\}$ (here θ_A denotes the zero element in A) are presented. It is shown that

(a) every jointly continuous bilinear map from $X \times Y$ into Z has a jointly continuous bilinear extension to $\tilde{X} \times \tilde{Y}$ in the case when X, Y, Z, $\tilde{X}$ and $\tilde{Y}$ are Hausdorff linear spaces (by this result it is easy to show that the completion of any topological Hausdorff algebra with a jointly continuous multiplication is a complete Hausdorff algebra with a jointly continuous multiplication);

(b) every locally m-convex (complete locally m-convex) Hausdorff algebra is topologically isomorphic to the dense subalgebra of the projective limit of Banach algebras;

(c) the quasi–inversion in every locally m-convex Hausdorff algebra is continuous;

(d) the spectrum of every element of any complex locally m-convex Hausdorff algebra is not empty and

(e) the multiplication in $A[X]$ (the algebra of polynomials with coefficients in a locally convex algebra A endowed with the direct sum topology) is jointly continuous, if the multiplication in A is jointly continuous.

Examples of such topological linear spaces X, which have (or have not) the so-called "countable neighbourhood property" (i.e., for any sequence (U_n) of neighbourhoods of zero in X there is a sequence (ϱ_n) of positive numbers such that the intersection of all sets $\varrho_n U_n$ is a neighbourhood of zero in X) are introduced and topological algebras (A,τ), which are a topological semidirect product of his two-sided ideal C and subalgebra B (then $A = C + B$, $C \cap B = \{\theta_A\}$ and the map $(c,b) \mapsto c+b$ is a homeomorphism from $(C, \tau \cap C) \times (B, \tau \cap B)$ onto (A,τ)) are described. It is shown, that every locally convex algebra, which is a topological semidirect product of a locally m-convex two-sided ideal and a locally m-convex subalgebra, is a locally m-convex algebra.

Properties connected with (continuous) linear multiplicative functionals on a locally convex algebra A are considered in Chapter 2. It is shown that every locally complete locally m-convex Hausdorff algebra with a fundamental sequence of bounded sets is functionally bounded (i.e., every linear multiplicative functional χ on A has the property: $\chi(B)$ is bounded on all bounded subsets $B \subset A$) and that every locally m-convex Fréchet algebra A is functionally continuous (i.e., every linear multiplicative functional χ on A is continuous) if and only if all locally complete and locally m-convex Hausdorff algebras are functionally bounded. Moreover, it is shown that every nontrivial linear multiplicative functional Φ on $C(X,A)$ (the algebra of all continuous A-valued functions on X with point-wise algebraic operations) defines a point $x \in X$ and a nontrivial linear multiplicative functional ϕ on A such that $\Phi(f) = \phi(f(x))$ for each $f \in C(X,A)$ in the following cases: (a) X is a realcompact space and A is a metrizable algebra with unity and (b) X is a compact space and A is a locally convex Hausdorff algebra, which satisfies the strict Mackey condition (i.e., for each bounded subset $B \subset A$ there is an absolutely convex bounded subset $D \subset A$ such that $B \subset D$ and the Minkowski funktional p_D of D induces the original topology on B). Similar descriptions of nontrivial linear multiplicative functionals on several other algebras of vector-valued functions are given, too.

Locally m-convex inductive topologies on countable inductive limits of locally m-convex algebras are considered in Chapter 3. It is shown that the finest locally convex topology on a countable inductive limit A of seminormed algebras A_n (or commutative locally m-convex algebras A_n, which satisfy the countable neighbourhood condition) is locally m-convex and every countable inductive limit of locally convex algebras A_n, which satisfy the countable neighbourhood condition, is a locally convex algebra, if every A_n is continuously embedded into A_{n+1} and all the inclusions $A_n \to A$ are continuous. Conditions which yield that the inductive limit of Moscatelli type of locally convex (locally m-convex) algebras is again a locally convex locally m-convex) algebra, are considered separately.

Several results presented in this paper are known, but reproved here. Some of these hold also in the locally pseudoconvex or locally m-pseudoconvex case.

Reviewer: Mati Abel (Tartu)

MSC:

46H05 General theory of topological algebras
46A04 Locally convex Fréchet spaces and (DF)-spaces
46E25 Rings and algebras of continuous, differentiable or analytic functions
46H20 Structure, classification of topological algebras

Keywords:

locally convex algebras