

Heath-Brown, D. R.

The largest prime factor of $X^3 + 2$. (English) Zbl 1023.11048
Proc. Lond. Math. Soc., III. Ser. 82, No. 3, 554-596 (2001).

Let f be an irreducible polynomial with positive leading coefficient, and define $P(x; f)$ to be the largest prime divisor of $\prod_{n \leq x} f(n)$. C. Hooley [J. Reine Angew. Math. 303/304, 21–50 (1978; [Zbl 0391.10028](#))] gave a proof that $P(x, X^3 + 2) > x^{31/30}$ provided one assumes “Hypothesis R^* ”, a best possible estimate for short Ramanujan-Kloosterman sums. In this paper, the author proves the unconditional estimate

$$P(x, X^3 + 2) > x^{1+\varpi} \quad \text{with} \quad \varpi = 10^{-303}.$$

Although the constant ϖ is very small, this result is significant because it is the first unconditional result of the form $P(x, f) > x^{1+\delta}$ for a polynomial of f of degree exceeding 2.

Problems involving $P(x, f)$ were first considered by Chebyshev, who sketched a proof that $P(x, X^2 + 1)/x \rightarrow \infty$. In this paper, the author develops a novel variant of Chebyshev’s method; this variant method is quite likely to be useful for other problems of Chebyshev type. The original Chebyshev method requires non-trivial estimates of appropriate error terms summed over prime arguments. In the variant, the error terms are summed over “smooth” arguments; i.e., over numbers with no large prime factors, and one has considerable freedom about choosing the arguments. The resulting error terms in this paper are estimated via a q -analogue of van der Corput’s method. The basic principles of this method were first sketched in the author’s paper “Hybrid bounds for L -functions: a q -analogue of van der Corput’s method and a t -analogue of Burgess’ method”, Recent progress in analytic number theory, (ed. H. Halberstam and C. Hooley, Academic Press, London, Vol. 1 (Durham 1979)), 121–126 (1981; [Zbl 0457.10021](#)).

Reviewer: [Sidney W. Graham \(Mount Pleasant\)](#)

MSC:

- [11N32](#) Primes represented by polynomials; other multiplicative structures of polynomial values
- [11L07](#) Estimates on exponential sums
- [11N36](#) Applications of sieve methods

Cited in **4** Reviews
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irreducible polynomial with positive leading coefficient; largest prime divisor; unconditional estimate

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