The notion of an invariant is certainly one of the most general and fundamental concepts in mathematics, since the construction of invariants is inevitably required whenever it comes to classify mathematical objects of some well-defined class. Actually, the origins of invariant theory can be traced back to the early 19th century, in particular to the systematic study of quadratic forms by Lagrange, Gauss, Cauchy, and Jacobi, on the one hand, and to the development of projective geometry by Poncelet, Möbius, Chasles, Steiner, Plücker, and Schläfli, on the other hand. Invariant theory as such arose in the middle of the 19th century, when the study of the general problem of constructing invariants of systems of forms of arbitrary degree was initiated, in close connection with the theory of determinants. At this stage, the most propelling contributions to the creation of invariant theory were made by G. Boole, O. Hesse, G. Salmon, J. Sylvester, A. Cayley, Ch. Hermite, G. Eisenstein, and other great mathematicians of that time. Then the baton was passed to S. Aronhold, F. Brioschi, A. Clebsch, P. Gordan, A. Capelli, S. Lie, and F. Klein, whose efforts were concentrated on the development of formal algebraic methods for constructing invariants and finding, in various concrete cases, a finite generating set and defining relations for a respective algebra of invariants. This first period in the history of invariant theory, the so-called “naive period” (or “pre-Hilbert period”), culminated with the discovery of the “symbolic method” which in theory allowed the computation of all respective invariants by an almost purely mechanical process. Complete results, however, were obtained only in a few simple cases, mainly for binary forms, as the actual computations by means of the symbolic method led to enormous labor, disproportionate with the interest of the outcome, especially in a period when all calculations had to be done by hand. Nevertheless, invariant theory was considered to be of great significance, during that period, because it was understood as being nearly identical to projective geometry. This point of view was most consistently and clearly expounded in F. Klein’s famous “Erlanger Program” in 1872.

Anyway, with regard to the limit of applicability of the symbolic method in invariant theory, the next problem was the search for “fundamental systems” of invariants, i.e., finite sets of invariants such that any invariant would be a polynomial expression in the fundamental invariants. The existence of such fundamental systems was proved, under certain assumptions, by D. Hilbert in 1890, in an ingenious paper which made him famous overnight and which may be considered as the first paper in modern algebra, due to its conceptual approach and methods. But Hilbert’s spectacular results also spelled the doom of invariant theory in those days, which was left with no more big problems to solve and soon sank into oblivion. Well, invariant theory has been pronounced dead several times, in the sequel, and like the phoenix it has been again and again rising from its ashes. The first revival was prompted by the works of I. Schur, H. Weyl, and B. Cartan on the global theory of semi-simple Lie groups and their representations in the 1930’s, from which it was realized that classical invariant theory was really a special case of that new theory.

But again a seeming lack of challenging problems was probably the reason why important new developments did not occur after the publication of H. Weyl’s famous book “The classical groups, their invariants and representations” (2nd edition Princeton, N.J. 1946; reprint 1997). This second period of stagnation of invariant theory ended abruptly in the late 1950’s, when M. Nagata constructed a counter-example to the 14th Hilbert problem, and the third golden age of invariant theory dawned with the publication of M. Mumford’s epoch-making book “Geometric invariant theory” (1965; Zbl 0147.39304). In fact, Mumford realized that invariant theory provided him with some of the tools he needed for his solution of the “moduli problem” for algebraic curves and principally polarised abelian varieties, and that some essential ideas and techniques for constructing algebraic orbit spaces (geometric quotients) lay buried in D. Hilbert’s brilliant (and long forgotten) papers on invariant theory from the 1890’s. In his book “Geometric invariant theory”, D. Mumford (loc. cit.) has radically modernized and greatly generalized Hilbert’s original ideas, using the language of modern algebraic geometry, as well as important results by Chevalley, Nagata, Grothendieck, Iwahori, Tate, Tits, and himself. In the past forty years, invariant theory, especially geometric invariant
theory, has established itself as a central topic of algebraic geometry and general classification theory in mathematics.

Thus invariant theory has a very long, fascinating and somewhat unique history, which has seen alternating periods of flourishing and stagnation, changes in its formulation, and varying fields of application. In these days, invariant theory stands there as an important part of the general theory of (algebraic) transformation groups, and as more momentous than ever in mathematics and physics.

The main purpose of the book under review is to give a concise and self-contained exposition of the main ideas of classical and modern invariant theory, with a special emphasis on the more recent (algebro-) geometric aspects of the subject, but basically following the historical path. According to the troubled history of invariant theory, the comparatively low number of experts and active researchers in the field, at least so in the “post-Hilbert period”, and due to the fact that invariant theory has not been a preferential teaching subject at universities, during the last century, there are not many accessible introductory textbooks covering invariant theory in its different aspects. Most classical texts do not cover the basics of Mumford’s geometric invariant theory, of course, and Mumford’s pioneering text “GIT” is just too advanced and difficult for beginners. Among the more recent texts on invariant theory introducing various aspects of the subject, there are just the books: “Invariant theory, old and new” by J. A. Dieudonné and J. B. Carrel from 1971 (New York and London 1971; Zbl 0258.14011), “Geometrische Methoden in der Invariantentheorie” by H. Kraft (Aspects Math. D1, Braunschweig und Wiesbaden, 1984; Zbl 0569.14003), “Algebraic transformation groups” by H. Kraft, P. Slodowy and T. A. Springer (DMV Seminar, 13; 1989; Zbl 0682.00008), and the encyclopedic survey “Invariant theory” by E. G. Vinberg and V. L. Popov [in: Algebraic geometry, IV, Encycl. Math. Sci. 55, 123-278 (1994); translation from Itogy Nauki Tekh., Ser. Sovrem., Probl. Mat., Fundam. Napravljeniya 55, 137-309 (1989; Zbl 0789.14008)].

Now there is also the book under review, based on several graduate courses taught by the author in the 1990’s at Seoul National University, the University of Michigan, and at Harvard University. As the author points out in the preface to his book, this text is literally intended to motivate and prepare a beginner to study modern invariant theory more thoroughly, especially with a view towards algebraic geometry (moduli problems) and differential geometry (Lie groups). The essential novelty offered by this text, among the very few comparable introductions to modern invariant theory, is the large number of examples and applications. Also, for the first time in an introductory text of geometric invariant theory, the author has included a thorough study of linearizations of group actions on varieties, stability properties and geometric quotients, and torus actions on affine spaces. A particular feature is given by the numerous concrete examples from classical projective geometry, providing a beautiful illustration of the abstract methods in geometric invariant theory and linking the different aspects of the whole subject.

As to the contents of the book, the text is subdivided into twelve chapters which lead the reader systematically from classical invariant theory to the fundamental concepts of modern geometric invariant theory, basically along the historical line of development described above. Here is a list of the single chapters, with keywords indicating their contents:

1. The symbolic method (first examples, the polarization and restitution process, bracket functions);
2. The first fundamental theorem (Cayley’s omega-process, Grassmann varieties, the straightening algorithm);
3. Reductive algebraic groups (the Gordan-Hilbert theorem, H. Weyl’s unitary trick, affine algebraic groups, Nagata’s theorem);
4. Hilbert’s fourteenth problem (the problem, Weitzenböck’s theorem, Nagata’s counter-example to Hilbert’s 14th problem);
5. Algebra of covariants (examples of covariants, covariants of an action, linear representations of reductive groups, dominant weights, the Cayley-Sylvester formula, standard tableaux);
6. Quotients (categorical and geometric quotients, rational quotients, Rosenlicht’s theorem, examples);
7. Linearizations of actions (linearized line bundles, existence of linearizations, the linearization of a group action);
8. Stability (stable points with respect to a linearized bundle, quotients as quasi-projective varieties, examples of quotients);
9. Numerical criterion of stability (Hilbert-Mumford stability and Kempf stability);
10. Projective hypersurfaces (invariants of non-singular projective hypersurfaces, binary forms, plane
cubics and cubic surfaces);
11. Configurations of linear subspaces (diagonal actions of $SL_{n+1}$ on products of Grassmannians, stable configurations, configurations of points in projective space, configurations of points in projective 3-space);
12. Toric varieties (torus actions on affine space, fans, toric varieties, examples).

Each chapter comes with its individual bibliographical notes for further reading, followed by a set of (quite challenging) exercises. Although many of the exercises are highly demanding, no hints for solution are offered, which might turn out to be pretty hard for beginners in the field. On the other hand, there are those many beautiful examples scattered throughout the text, helping the reader to grasp the fascination of (old and new) invariant theory.

The exposition is very systematic, lucid and sufficiently detailed. It certainly transpires the author’s passion, versatility, all-round knowledge in mathematics, and mastery as both an active researcher and devoted teacher. It is very gratifying to see this beautiful, modern and still down-to-earth introduction to classical and contemporary invariant theory at the public’s disposal, after a long period of craving for suitable standard texts in this literarily somewhat neglected field.

The book assumes only minimal prerequisites for graduate students: a basic knowledge of algebraic varieties, a profound knowledge of multilinear algebra, and some rudiments of the theory of linear representations of groups. Everything else, including the necessary facts from the theory of linear algebraic groups, is sufficiently explained in the course of the text. In this regard, the present book is really nearly self-contained.

Finally, as to the merely computational aspects of invariant theory, it should be remarked that those are not particularly stressed in this approach. The interested reader might find it very useful to consult the recent book “Computational invariant theory” by H. Derksen and G. Kemper [Encyclopaedia of Math. Sci., Invariant theory and algebraic transformation groups, 130 (1) (2002; Zbl 1011.13003)] for additional and parallel reading. In fact, the development of computational commutative algebra has provided another decisive impulse to revive the classical (algorithmic) methods of invariant theory, with numerous important applications in various fields, including the invariant theory of reductive groups and the construction of moduli spaces in geometry, and these new aspects of constructive invariant theory and its applications are comprehensively expounded in that just as recent reference book.

Reviewer: Werner Kleinert (Berlin)

MSC:

13A50 Actions of groups on commutative rings; invariant theory
15A72 Vector and tensor algebra, theory of invariants
13-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to commutative algebra
14L24 Geometric invariant theory
14-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to algebraic geometry
15-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to linear algebra

Keywords:
invariant theory; geometric invariant theory; geometric quotients; orbit spaces; Hilbert’s fourteenth problem; Lie groups; GIT; configurations of linear subspaces; linearizations of actions; classification theory; stability

Cited in 2 Reviews
Cited in 124 Documents