Bouchut, F.; **Desvillettes, L.**

On two-dimensional Hamiltonian transport equations with continuous coefficients. (English)

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According to the theory of characteristics, the linear transport equation

$$\partial_t u(t,x) + a(t,x)\cdot\nabla_x u(t,x) = 0, \quad u(0,x) = u^-(x), \tag{1}$$

where $t\in\mathbb{R}$, $x\in\mathbb{R}^N$, $u^0:\mathbb{R}^N\to\mathbb{R}$, $a:\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ and $u:\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$, is related to the system of ordinary differential equations

$$\frac{dX}{ds}(s) = a(s,X(s)), \quad X(t) = x,$$

via the relation $u(t,x) = u^0(X(t,x))$. This system admits a unique (local) solution of class C^1 as soon as $a\in C^1(\mathbb{R}\times\mathbb{R}^N;\mathbb{R}^N)$. The authors prove here that the regularity of a can even be relaxed (generically) to a only continuous when $N=2$ and a does not depend on t (and $\operatorname{div}_x a = 0$), that is, for autonomous two-dimensional Hamiltonian fields

$$a(x_1,x_2) = \left(-\frac{\partial H}{\partial x_2}(x_1,x_2), \frac{\partial H}{\partial x_1}(x_1,x_2)\right), \quad H\in C^1(\mathbb{R}^2;\mathbb{R}).$$

This leads to the Hamiltonian system

$$\begin{aligned}&\frac{dX_1}{dt}(t) = -\frac{\partial H}{\partial x_2}(X_1(t),X_2(t)), \frac{dX_2}{dt}(t) = \frac{\partial H}{\partial x_1}(X_1(t),X_2(t)),\\ &(X_1(0),X_2(0))) = (x_1^0,x_2^0).\end{aligned} \tag{2}$$

In Section 2, under the assumption that $H(x^0)$ is not a critical value of H, the uniqueness of the solutions to system (2) is proved. Then, a generic condition on a such that there is uniqueness of the solutions to system (2) for almost every $x^0\in\mathbb{R}^2$ is presented. Finally, an analogous result of uniqueness for the transport equation (1) is proved.

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