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A rigid analytic Gross-Zagier formula and arithmetic applications. (With an appendix by B. Edixhoven). (English) Zbl 1029.11027

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Let f be a newform of weight 2 and squarefree level N. Its Fourier coefficients generate a ring $\mathcal{O}_f$ whose fraction field K_f has finite degree over $\mathbb{Q}$. Fix an imaginary quadratic field K of discriminant prime to N, corresponding to a Dirichlet character ε. The L-series $L(f/K,s)=L(f,s)L(f\otimes\varepsilon,s)$ of f over K has an analytic continuation to the whole complex plane and a functional equation relating $L(f/K,s)$ to $L(f/K,2-s)$. Assume that the sign of this functional equation is 1, so that $L(f/K,s)$ vanishes to even order at $s=1$. This is equivalent to saying that the number of prime factors of N which are inert in K is odd. Fix any such prime, say p.

The field K determines a factorization $N=N^+N^-$ of N by taking N^+, resp. N^- to be the product of all the prime factors of N which are split, resp. inert in K. Given a ring-class field extension H of K of conductor c prime to N, write H_n for the ring-class field of conductor cp^n.

Let J be the Jacobian of X, $\mathcal{J}_n$ the Néron model of J over H_n, and Φ_n the group of connected components at p of $\mathcal{J}_n$. More precisely, $\Phi_n:=\bigoplus_{\mathfrak{p}|p}\Phi_{\mathfrak{p}}$, where $\Phi_{\mathfrak{p}}$ is the group of connected components of the fiber at $\mathfrak{p}$ of $\mathcal{J}_n$ and the sum is extended over all primes $\mathfrak{p}$ of H_n above p. Define a Heegner divisor $\alpha_n:=(P_n)-(w_NP_n)$, where w_N is the Atkin-Lehner involution denoted $w_{N^+p,N^-/p}$. We view α_n as an element of $\mathcal{J}_n$, and let $\overline{\alpha}_n$ be its natural image in Φ_n. We have found that the position of $\overline{\alpha}_n$ in Φ_n is encoded in the special values of the L-functions attached to cusp forms of weight 2 on X twisted by characters χ of $\Delta:=\mathrm{Gal}(H/K)$.

More precisely, observe that the Galois group $\mathrm{Gal}(H_n/K)$ acts on $J(H_n)$ and on $\mathcal{J}_n$. Since the primes above p are totally ramified in H_n/H, the induced action on Φ_n factors through Δ. Define $e_\chi:=\sum_{g\in\Delta}\chi^{-1}(g)g\in\mathbb{Z}[\chi][\Delta]$, and let $\overline{\alpha}_n^\chi:=e_\chi\overline{\alpha}_n$. The ring $\mathbb{T}$ generated over $\mathbb{Z}$ by the Hecke correspondences on X acts in a compatible way on $J(H_n)$, $\mathcal{J}_n$ and Φ_n. Write $\varphi_f:\mathbb{T}\to\mathcal{O}_f$ for the homomorphism associated to f by the Jacquet-Langlands correspondence, and let $\pi_f\in\mathbb{T}\otimes K_f$ be the idempotent corresponding to φ_f. Fix $n_f\in\mathcal{O}_f$ so that $\eta_f:=n_f\pi_f$ belongs to $\mathbb{T}\otimes\mathcal{O}_f$, and define $\overline{\alpha}_n^{f,\chi}:=\eta_f\overline{\alpha}_n^\chi$.

The group Φ_n is equipped with a canonical monodromy pairing $[\ ,\ ]_n:\Phi_n\times\Phi_n\to\mathbb{Q}/\mathbb{Z}$, which we extend to a Hermitian pairing on $\Phi_n\otimes\mathcal{O}_f[\chi]$ with values in $K_f[\chi]/\mathcal{O}_f[\chi]$, denoted in the same way by abuse of notation. Our main result is:

Theorem A. Suppose that χ is a primitive character of Δ. Then

$$[\overline{\alpha}_n^\chi,\overline{\alpha}_n^{f,\chi}]_n=\frac{1}{e_n}\frac{L(f/K,\chi,1)}{(f,f)}\sqrt{d}\cdot u^2\cdot n_f\pmod{\mathcal{O}_f[\chi]},$$

where (f,f) is the Petersson scalar product of f with itself, and d denotes the discriminant of $\mathcal{O}$.

The proof is based on Grothendieck's description of Φ_n, on the work of Edixhoven on the specialization map from $\mathcal{J}_n$ to Φ_n given in the appendix of this paper, and on a slight generalization of Gross' formula for special values of L-series (which we assume in this paper and which will be contained in [*H. Daghigh*, Ph.D. thesis]). Theorem A can be viewed as a p-adic analytic analogue of the Gross-Zagier formula, and it was suggested by the conjectures of Mazur-Tate-Teitelbaum type formulated in [*M. Bertolini* and *H. Darmon*, Invent. Math. 126, 413-456 (1996; Zbl 0882.11034)]. It is considerably simpler to prove than the Gross-Zagier formula, as it involves neither derivatives of L-series nor global heights of Heegner points.

The above formula has a number of arithmetic applications. Let A_f be the Abelian variety quotient of J associated to φ_f by the Eichler-Shimura construction. Following the methods of Kolyvagin, we can use the Heegner points α_n to construct certain cohomology classes in $H^1(H,(A_f)_{e_n})$, whose local behaviour is related via Theorem A to $L(A_f/K,\chi,1)=\prod_\sigma L(f^\sigma/K,\chi,1)$, where σ ranges over the set of embeddings of K_f in $\overline{\mathbb{Q}}$. This can be used to study the structure of the χ-isotypical component $A_f(H)^\chi:=e_\chi A_f(H)\subset A_f(H)\otimes\mathbb{Z}[\chi]$ of the Mordell-Weil group $A_f(H)$. In particular, we show:

Theorem B. If $L(A_f/K,\chi,1)$ is nonzero, then $A_f(H)^\chi$ is finite.

When $\chi=\overline{\chi}$, this result also follows from the work of Gross-Zagier and Kolyvagin-Logachev, but if χ is nonquadratic the previous techniques cannot be used to study these questions.

Theorem B allows us to control the growth of Mordell-Weil groups over anticyclotomic $\mathbb{Z}_\ell$-extensions, addressing a conjecture of Mazur. Let f and K be as at the beginning of this section. Let $\ell_1, \dots, \ell_k$ be primes not dividing N, and let K_∞ denote the compositum of all the ring-class field extensions of K of conductor of the form $\ell_1^{n_1} \dots \ell_k^{n_k}$, where $n_1, \dots, n_k$ are nonnegative integers. Thus, the Galois group of K_∞/K is isomorphic to the product of a finite group by $\mathbb{Z}_{\ell_1} \times \cdots \times \mathbb{Z}_{\ell_k}$.

Theorems A and B provide a technique to study "analytic rank-zero situations" in terms of Heegner points of conductor divisible by powers of a prime p of multiplicative reduction for A_f and inert in K. What makes this possible, ultimately, is a "change of signs" phenomenon: If $L(f/K, s)$ vanishes to even order, and χ is an anticyclotomic character of conductor cp^n with c prime to N, then $L(f/K, \chi, s)$ vanishes to odd order, and there are Heegner points on A_f defined over the extension cut out by χ. The previous applications of the theory of Heegner points, such as the analytic formula of Gross-Zagier and the methods of Kolyvagin, occur in situations where $L(f/K, s)$ and $L(f/K, \chi, s)$ both vanish to odd order.

Reviewer: O.Ninnemann (Berlin)

MSC:

11G40 L-functions of varieties over global fields; Birch-Swinnerton-Dyer conjecture

11F67 Special values of automorphic L-series, periods of automorphic forms, cohomology, modular symbols

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