Benedetto, John J.; **Treiber, Oliver M.**

Wavelet frames: Multiresolution analysis and extension principles. (English) Zbl 1036.42032

Debnath, Lokenath, Wavelet transforms and time-frequency signal analysis. Boston, MA: Birkhäuser (ISBN 0-8176-4104-1/hbk). Applied and Numerical Harmonic Analysis, 3-36 (2001).

Frame multiresolution analysis was introduced by *J. J. Benedetto* and *S. Li* [Appl. Comput. Harmon. Anal. 5, 389–427 (1998; Zbl 0915.42029)]. For $f \in L^2(\mathbb{R})$, let $(T_a f)(x) = f(x-a)$ and $(D_a f)(x) = a^{-1/2} f(\frac{x}{a}), a \neq 0$. An FMRA for $L^2(\mathbb{R})$ consists of a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}} \subseteq L^2(\mathbb{R})$ and a function $\phi \in V_0$ such that (i) $\cdots V_{-1} \subseteq V_0 \subseteq V_1 \cdots$, (ii) $\overline{\cup_j V_j} = L^2(\mathbb{R})$ and $\overline{\cap_j V_j} = 0$, (iii) $f \in V_j \Leftrightarrow [t \to f(2t)] \in V_{j+1}$, (iv) $f \in V_0 \Rightarrow T_k f \in V_0$, $\forall k \in \mathbb{Z}$, and (v) $\{T_k \phi\}_{k\in\mathbb{Z}}$ is a frame for V_0.

The purpose of an FMRA is to construct wavelet frames for $L^2(\mathbb{R})$, i.e., frames of the type $\{D_{2^j} T_k \psi\}_{j,k\in\mathbb{Z}}$; the obvious idea is to proceed as in the construction of an orthonormal basis via an MRA. Let W_j denote the orthogonal complement of V_j in V_{j+1}. The main question is to find a wavelet ψ such that $\{T_k\psi\}_{k\in\mathbb{Z}}$ is a frame for W_0; this implies that $\{D_{2^j} T_k \psi\}_{j,k\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. Here the authors prove that the existence of such a function ψ depends solely on the "size" of the set $\Gamma := \{x \in [0,1] : \Phi(2x) = 0, \Phi(x) > 0, \Phi(x + \frac{1}{2}) > 0\}$, where $\Phi(x) = \sum_{k\in\mathbb{Z}} |\hat{\phi}(x+k)|^2$. In fact, there exists a function $\psi \in L^2(\mathbb{R})$ such that $\{T_k\psi\}_{k\in\mathbb{Z}}$ is a frame for W_0 if and only if Γ has vanishing Lebesgue measure. In case Γ has vanishing Lebesgue measure, the authors also show how to define a suitable function ψ.

It is explained how overcompleteness of frames implies robustness against noise that appear for example via transmission. Furthermore, a new direct proof of Ron and Shen's Unitary Extension Principle is given.

For the entire collection see [Zbl 0996.00017].

Reviewer: Ole Christensen (Lyngby)

MSC:

42C40 Nontrigonometric harmonic analysis involving wavelets and other special systems

65T60 Numerical methods for wavelets

Keywords:

wavelet frames; frame multiresolution analysis; unitary extension principle