The theory of quantum structures was introduced in the beginning of the 1930s by Birkhoff and von Neumann as mathematical foundations of quantum physics. “New physics” of those days, quantum mechanics, showed that events connected with any measurement process of quantum observables do not fulfill the axiomatic of probability theory and statistics. During the last 70 years, many approaches to axiomatize quantum mechanics have been developed. Besides algebraic or convex analysis the most important base is concerned with the realm of the Hilbert space.


The present monograph extends the latest and very important results concentrated around measure theory on quantum structures. Whereas the cornerstone of the monograph by Dvurečenskij (1993), centered around the Gleason theorem and its application in the world of the logic of a Hilbert space or of an inner product space, the book under the review is focusing on the extension of Gleason’s theorem for von Neumann algebras, and therefore, it is very close to direct applications to mathematical physics.

The monograph consists of 11 chapters, bibliography, and index. The introductory chapter shortly presents the contents of the book. The second chapter gives elements of $C^*$-algebras, von Neumann algebras, Jordan and ordered structures, and operator algebras.

The cornerstone of the book as well as of the theory under review is the Gleason theorem, studied in Chapter 3 [A. M. Gleason, J. Math. Mech. 6, 885–893 (1957; Zbl 0078.28803)]. It says that every $\sigma$-additive state $\mu$ on the projection lattice $P(H)$ of a separable Hilbert space $H$ is in a one-to-one correspondence with a von Neumann operator $T$ such that $\mu(P) = \text{tr}(TP), P \in P(H)$. It is fascinating that the heart of the proof is lying in the three-dimensional Hilbert space $\mathbb{R}^3$. The original proof was very complicated, and it was only in the middle of the 1980s when R. Cooke, M. Keane, and W. Moran [Math. Proc. Camb. Philos. Soc. 98, 117–128 (1985; Zbl 0575.46051)] gave a more elementary proof, that is presented here. Every state (or charge) can be studied by its frame-function (a function defined on the unit-sphere of $H$). If we extend the notion of state to a charge, then in every $P(H)$, dim $H < \infty$, there is an unbounded charge. Then S. Dorofeev and A. Sherstnev [Sov. Math. 34, No. 4, 25–31 (1990; Zbl 0739.46047)] showed the unexpected result that in an infinite dimensional Hilbert space, every charge (frame) is bounded (Theorem 3.3.5).

In 1987, J. Hamhalter and P. Pták [Bull. Lond. Math. Soc. 19, 259–263 (1987; Zbl 0601.46027)] proved the interesting result that a real separable inner product space $S$ is complete if and only if the system $F(S)$ of all closed subspace $M$ of $S$ such that $M^\perp \perp = M$ admits a $\sigma$-additive state. This was a first measure-theoretic completeness criterion. This important result initiated the study of completeness criteria during the last 15 years (see Dvurečenskij, 1993). Chapter 4 is devoted to completeness criteria. These criteria often use algebraic criteria (Ameniya-Araki criterion saying that $S$ complete iff $F(S)$ is orthomodular).

In the book under review the author mainly concentrates to state criteria which were proved by him and also by other authors, e.g. A. Dvurečenskij [Ann. Inst. Henri Poincaré, Phys. Théor. 62, No. 4, 429–438 (1995; Zbl 0835.46017)].

Chapter 5 is devoted to the generalized Gleason theorem. It deals with an answer to the problem posed by G. W. Mackey who asked: Does any bounded finitely additive measure on the projection lattice $P(M)$ of a von Neumann algebra $M$ extend to a bounded linear functional on $M$? This problem was solved for many years. In this chapter, the author presents the generalized Gleason theorem which says that every bounded Banach-valued measure on $P(M)$ of a von Neumann algebra $M$ without direct summand of type $I_2$ extends uniquely to a bounded linear operator $T$ from $M$ to the Banach space. This is a result of L.
Chapter 6 is devoted to studying basic principles of quantum measure theory. It shows when a measure on $P(H)$ is bounded and it also gives some generalizations of known results of classical measure theory like the Vitali-Hahn-Saks theorem, Egorov theorem, Yosida-Hewitt decomposition, and the Lyapunov theorem.

The next chapter gives some applications of Gleason’s theorem. The first deals with the Gleason theorem for multimeasures (decoherence functionals, history approach), dynamical aspects, and problem of hidden variables.

Chapter 8 presents some problems of orthomorphisms of projections. This is a generalized version of the Wigner theorem. In particular, it describes the structure of completely additive orthomorphisms between Hilbert space projection lattices and gives the well known Wigner unitary-antiunitary theorem as a byproduct.

Chapter 9 describes the restriction and extension properties of states on $C^\ast$-algebras. The author derives some results on the determinacy of the system of orthogonal pure states by biorthogonal systems of elements in the $C^\ast$-algebra. Then he concentrates on the extension of measures.

Chapter 10 deals with Jauch-Piron states. Such states have a kernel that is closed under forming finite suprema of projections. The aim is to show how topological and algebraic aspects of the projection lattice are closely related, and it gives new insight in axioms of the operator-algebraic approach to quantum mechanics.

The last Chapter 10 focusses on problems of independence of quantum systems like von Neumann and $C^\ast$-algebras. Independence is a well-known notion in classical probability theory, and in this chapter some applications like $C^\ast$-independence and $W^\ast$-independence are studied.

Finally, the References contain 334 items.

The monograph under review collects many important and highly nontrivial results and efforts of many authors. It is important to recall that the basic material is based on the research done by J. Hamhalter who is influencing many researchers in this field. The style is very fresh and the author is permanently keeping an eye on the reader’s trip through the book, and I highly recommend this book to students and experts interested in operator algebras, noncommutative measure theory and mathematical foundations of quantum physics. The monograph is a welcome addition to the quantum structures realm.

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MSC:

81-02 Research exposition (monographs, survey articles) pertaining to quantum theory
81P10 Logical foundations of quantum mechanics; quantum logic (quantum-theoretic aspects)
46L30 States of selfadjoint operator algebras
46L10 General theory of von Neumann algebras
46L60 Applications of selfadjoint operator algebras to physics
81P15 Quantum measurement theory, state operations, state preparations
03G12 Quantum logic
06C15 Complemented lattices, orthocomplemented lattices and posets

Keywords:
von Neumann algebra; Gleason theorem; quantum measure theory; projection lattice; Jauch-Piron state; independence; $C^\ast$-algebras; Jordan structure; ordered structure; operator algebras