Tate, John

A review of non-Archimedean elliptic functions. (English) Zbl 1071.11508

Coates, John (ed.) et al., Elliptic curves, modular forms, and Fermat's last theorem. Proceedings of the conference on elliptic curves and modular forms held at the Chinese University of Hong Kong, December 18-21, 1993. Cambridge, MA: International Press (ISBN 1-57146-026-8/hbk). Ser. Number Theory 1, 162-184 (1995).

This expository article consists of two parts. It is written mostly in a nicely elementary way – which will please beginners and students in the field – but requires more background in some cases (especially at the end of the text).

The first part, "Rational points on elliptic curves over complete fields", is in fact a 1959 manuscript. It explains what is now known as Tate's curve. Motivated by classical series expansions for elliptic curves over the complex numbers, Tate constructs over a complete field k an elliptic curve E_t (where t is some element of k satisfying $0 < |t| < 1$), together with an isomorphism $k^*/t^{\mathbb{Z}} \to E_t(k)$ of abelian groups.

At this level, the theory is quite simple. One uses the known expansions over $\mathbb{C}$ and proves some convergence properties to get a parametrization $k^* \to E_t(k)$. One deduces from the known functional equations over $\mathbb{C}$ equalities of power series, which then give the functional equations needed as functions on k (or k^*). Two points really need a new proof: the surjectivity of the parametrization $k^* \to E_t(k)$ and the computation of the kernel. Another, maybe simpler, proof of the surjectivity is given in the second part of the paper.

The second part of the paper is more recent. Various topics are covered: theta functions from the "rigid analytic" point of view, isogenies between E_t's, an isogeny invariant of an elliptic curve which appears in a theorem by *R. Greenberg* and *G. Stevens* [Invent. Math. 111, No. 2, 407–447 (1993; Zbl 0778.11034)]. It also gives applications to points of finite order on Tate's curves and to Serre's isogeny theorem for such. It ends with some remarks on modular curves where Tate's curve (on $\mathbb{Z}[\![q]\!]$) is used to study the modular curves at infinity, so as to define algebraically q-expansions for modular forms.

For the entire collection see [Zbl 0824.00025].

Reviewer: Antoine Chambert-Loir (MR1363501)

MSC:

11G07 Elliptic curves over local fields

14G20 Local ground fields in algebraic geometry