Mason, J. C.
The minimality properties of Chebyshev polynomials and their lacunary series. (English) [Zbl 1078.33007]

Let $T_n, U_n, V_n$ and $W_n$ be four kinds of Chebyshev polynomials

$$T_n(x) = \cos(n\theta), \quad U_n(x) = \frac{\sin((n + 1)\theta)}{\sin \theta},$$

$$V_n(x) = \frac{\cos((n + \frac{1}{2})\theta)}{\cos \left( \frac{\theta}{2} \right)}, \quad W_n(x) = \frac{\sin \left( \frac{(n + \frac{1}{2})\theta}{\sin \left( \frac{\theta}{2} \right) \right)}{\sin \left( \frac{\theta}{2} \right)},$$

where $x = \cos \theta, 0 \leq \theta \leq \pi$. Define weight functions $w_p(x), -1 \leq x \leq 1$, for these polynomials as follows: if $1 \leq p < \infty$, then

$$w_p(x) = \begin{cases} (1 - x^2)^{-\frac{1}{p}} & \text{for } T_n, \\ (1 - x^2)^{\frac{1}{p}} & \text{for } U_n, \\ (1 + x)^{\frac{1}{p}}(1 - x)^{-\frac{1}{2}} & \text{for } V_n, \\ (1 - x)^{\frac{1}{p}}(1 + x)^{-\frac{1}{2}} & \text{for } W_n \end{cases}$$

and if $p = \infty$

$$w_\infty(x) = \lim_{p \to +\infty} (w_p(x))^{\frac{1}{p}}.$$

The author proves the following theorems that extend a number of known results on Chebyshev polynomials.

Chebyshev’s equioscillation theorem: Let $p_n$ be a polynomial, $\deg p_n = n$, and $f$ be continuous. For four cases of $w_\infty(x)$, the norm

$$\|f - p_n\| = \max_{-1 \leq x \leq 1} |w_\infty(x)(f(x) - p_n(x))|$$

is minimised if and only if $w_\infty(f - p_n)$ attains its maximum magnitude with alternating signs on at least $n + 2$ consecutive points of $[-1, 1]$.

The $L_p$ minimality property: The monic polynomials which are corresponding $T_n, U_n, V_n, W_n$ are the best $L_p$ approximations to zero on $[-1, 1]$ with respect to their $w_p(x), 1 \leq p \leq \infty$.

A sufficient interpolation condition for weight $L_1$ approximation: Let $f$ be a continuous function on $[-1, 1]$. A polynomial $p_n$, $\deg p_n = n$, is the best $L_1$ approximation of $f$ with respect to $w_1(x)$ if zeros of $f - p_n$ coincide with zeros of the relevant $n$-th Chebyshev polynomial corresponding to $w_1(x)$.

The best $L_p$ approximation by partial sums of lacunary series: Let $S^r_n$ ($r = 1, 2, 3, 4$) denote the sum of the first $n$ terms of the respective series:

(i) $S^1(x) \sim \sum_{k=1}^{\infty} a^k T_{k^r}(x)$,  
(ii) $S^2(x) \sim \sum_{k=1}^{\infty} a^k U_{k-1}(x)$,  
(iii) $S^3(x) \sim \sum_{k=1}^{\infty} a^k V_{\frac{1}{2}(k^r-1)}(x)$,  
(iv) $S^4(x) \sim \sum_{k=1}^{\infty} a^k W_{\frac{1}{2}(k^r-1)}(x)$.

Then $S^r_n$ are the minimax approximation to $S^r(r = 1, 2, 3, 4)$ with respect to the corresponding $w_\infty(x)$, given that $a$ is real, $b$ is an odd integer, and $|ab| \leq 1$. The variants of the last theorem are also proved for $p = 1$ and $1 < p < \infty$.

Reviewer: Aleksey A. Dovgoshey (Donetsk)

MSC:

33C45 Orthogonal polynomials and functions of hypergeometric type (Jacobi, Laguerre, Hermite, Askey scheme, etc.)
42C15 General harmonic expansions, frames
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References:

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