Sándor, Jozsef; Crstici, Borislav
Handbook of number theory II. (English) Zbl 1079.11001
Dordrecht: Kluwer Academic Publishers (ISBN 1-4020-2546-7/hbk). 637 p. (2004).

Similarly to the *Handbook of Number Theory* by *D. Mitrinović, J. Sándor* and *B. Crstici* (Kluwer) (1995; Zbl 0862.11001), it is the aim of this monograph to give, in an encyclopedic manner, surveys on some topics of Number Theory with many hints to the literature.

From the Preface: "The aim of this book is to systematize and survey in an easily accessible manner the most important results from some parts of Number Theory ... . Each chapter can be viewed as an encyclopedia of the considered field. ... This book focuses too, as the former volume, on some important arithmetic functions ..., such as Euler's totient $\varphi(n)$, ..., $\sigma(n)$ with the many old and new issues on Perfect Numbers; the Möbius function ... . The last chapter shows perhaps most strikingly the cross-fertilization of Number Theory with Combinatorics, Numerical mathematics, or Probability theory."

So, this monograph deals with topics from number theory, which we are going to try to describe now.

Perfect Numbers (some history; even perfect numbers; many results for odd perfect numbers). The chapter ends with generalizations of the notion of perfect number [multiply perfect, quasi-perfect, almost perfect, super-perfect, pseudo-perfect, unitary perfect, ...] and of amicable numbers. There are more than 300 references.

Generalizations and extensions of the Möbius function. This chapter is of a rather elementary nature. Many generalizations of the Möbius function (and of the notion of convolution) are given, including the Möbius function of arithmetical semigroups. There are more than 200 references.

The many facets of Euler's totient. There are connections with prime number theory (proof of the infinitude of primes by E. E. Kummer, the exact formula $\pi(n) = \sum_{2\le k\le n}[\frac{\varphi(n)}{k-1}]$ of Vassil'ev, and other such formulae), many identities (by Liouville, Cesáro, and others), Redmond's result

$$\prod_{n=1}^{\infty}\left(\frac{L_n}{\sqrt{5}}\cdot F_n\right)\frac{\varphi(n)}{n} = e^{-\frac{1+\sqrt{5}}{2\sqrt{5}}}.$$

Next, the author deals with enumeration problems, with Fourier coefficients of even functions (E. Cohen), with algebraic independence of certain arithmetical functions, with congruence properties (the Fermat-Euler theorem $a^{\varphi(m)} \equiv 1 \bmod m$, if $\gcd(a, m) = 1$, and many later results), Carmichael's function (minimal order, average order), iterates of φ, the behaviour in residue classes, for example

$$\#\{n \le x,\ \varphi(n) \equiv r \bmod m\} \sim c_r \cdot \frac{x}{\sqrt{\log x}}$$

for $m = 12, r = 4, 8$ (and further results), prime totatives, the function $n \mapsto n - \varphi(n)$, Euler's functions $\min\{k \ge 1,\ n \mid \varphi(k)\}$ and $\max\{k \ge 1,\ \varphi(k) \mid n\}$. Next there are results on equations of the type $\varphi(x + k) = \varphi(x)$, on the the equation $\varphi(x) = k$, on the solvability of equations involving φ and other arithmetical functions (for example the result of Erdős, that the system $\varphi(x) = \varphi(y),\ d(x) = d(y),\ \sigma(x) = \sigma(y)$ has infinitely many [non-trivial] solutions), results on the composition of φ with other functions, the distribution of totatives, ... Next, there are many results on cyclotomic polynomials (irreducibility, divisibility properties, the coefficients of these), and on matrices and determinants connected with φ. Finally, many generalizations of φ are dealt with. There is a most valuable list of nearly 500 references, from Gauss (1801), Dedekind (1857) up to the most recent results.

Special Arithmetic Functions. "The aim of this chapter is to study some other functions which ... are not so well-known, and are scattered in various fields of study ...."

Examples are the maximum (and minimum) exponent $H(n) = \max\{a_1, \dots, a_r\}$, where $n = \prod_\rho p_\rho^{a_\rho^{(n)}}$, the product of exponents $\beta(n) = a_1 \cdots a_r$, the function $n \mapsto a_\rho(n)$, the study of consecutive prime divisors of a number, the consecutive divisors of a number, Hooley's function

$$\Delta(n) = \max_{x \in \mathbb{R}} \#\{d \mid n,\ x < \log d \leq x + 1\},$$

and divisors in residue classes. There is a long subsection on arithmetic functions associated to the digits of a number, including q-additive and q-multiplicative functions. 360 references are given.

Stirling, Bell, Bernoulli, Euler and Eulerian Numbers. To describe the contents of this chapter in short, we quote from the introduction: "This Chapter is divided into two major parts: Stirling and Bell numbers, and Bernoulli, Euler and Eulerian numbers. These classical topics occur in practically every field of mathematics, in particular in combinatorial theory, finite difference calculus, numerical analysis, number theory, and probability theory. Our aim is to study the many aspects of these numbers, and to point out important connections or applications in number theory and related fields. ..." There are again nearly 500 references to the literature.

The reviewer thinks that this monograph is a great help to number theorists working in the fields mentioned above.

Reviewer: Wolfgang Schwarz (Frankfurt / Main)

MSC:

- 11-00 General reference works (handbooks, dictionaries, bibliographies, etc.) pertaining to number theory
- 11-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to number theory
- 11-02 Research exposition (monographs, survey articles) pertaining to number theory
- 11A25 Arithmetic functions; related numbers; inversion formulas
- 11B68 Bernoulli and Euler numbers and polynomials
- 11B73 Bell and Stirling numbers
- 11N37 Asymptotic results on arithmetic functions
- 11N64 Other results on the distribution of values or the characterization of arithmetic functions

Keywords:

arithmetic functions; Euler's φ–function; Möbiusfunction; perfect numbers; convolutions; Carmichael's function; composition of arithmetical functions; Stirling Numbers; Bell Numbers; consecutive divisors; special arithmetical functions