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Conics in the Grothendieck ring. (English) Zbl 1082.14022

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Let $K_0(\mathrm{Var}_k)$ denote the Grothendieck ring of algebraic k-schemes, with addition and multiplication given by disjoint sums and products, respectively. This paper computes the subring generated by the smooth conics $C \subset \mathbb{P}^2$, which are the 1-dimensional Severi–Brauer varieties. There is a technical assumption on the ground field k, but number fields, functions fields of complex surfaces, and more generally C_2-fields are allowed.

To describe the result, let G be a finite subgroup inside the 2-torsion of the Brauer group $\mathrm{Br}(k)$. Choose a basis $C_1, \dots, C_n \in G$ consisting of smooth conics, where G is regarded as vector space over the field with two elements, and let $C(G) = [C_1 \times \ldots \times C_n]$ be the class in the Grothendieck ring. Then the subring of the Grothendieck ring generated by smooth conics is the free abelian group generated by elements of the form $C(G) \cdot [\mathbb{P}^1]^m$, for varying G and m. There is also an explicit formula for multiplication.

The computation depends on another result of the paper, which characterizes when two schemes of the form $C_1 \times \ldots \times C_n$ and $C_1' \times \ldots \times C_{n'}'$, where all factors are smooth conics, have the same class in the Grothendieck ring. It turns out that this holds if and only if the following equivalent conditions hold: The two schemes are (1) birational, (2) stably birational, or (3) we have $n = n'$ and the subgroups of the Brauer groups generated by the factors $C_1, \dots, C_n$ and $C_1', \dots, C_n'$ are the same.

The proofs are based on work of *M. Larsen* and *V. A. Lunts* [Mosc. Math. J. 3, 85–95 (2003; Zbl 1056.14015)], and a nice geometric description of the Brauer product of two smooth conics C_1, C_2 as a subscheme of the Hilbert scheme of divisors on $C_1 \times C_2$ of bidegree (1,1).

Reviewer: Stefan Schröer (Düsseldorf)

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