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Zudilin, V. V.

On the functional transcendence of q-zeta values. (English. Russian original) Zbl 1093.11055

Math. Notes 73, No. 4, 588-589 (2003); translation from Mat. Zametki 73, No. 4, 629-630 (2003).

From the text: For any integer $k \geq 1$, the power series

$$\zeta_q(k+1) = \sum_{n=1}^{\infty} \sigma_k(n) q^n, \quad \sigma_k(n) = \sum_{d|n} d^k, \tag{2}$$

determines a q-extension of the value $\zeta(k+1)$ of the Riemann zeta-function *V. V. Zudilin* [Math. Notes 72, No. 6, 858–862 (2002); translation from Mat. Zametki 72, No. 6, 936–940 (2002; Zbl 1044.11066)]. Moreover, the series in (1) is also meaningful for $k = 0$. By virtue of the trivial estimates

$$\sigma_k(n) \leq n^k \sum_{d|n} 1 \leq n^{k+1)},$$

this series represents an analytic function inside the unit disc for each integer $k \geq 0$. The objective of this paper is to prove that the function $\zeta_q(k+1)$ is not algebraic for any $k \geq 1$. This (and even a stronger) result is well known for $\zeta_q(2)$, $\zeta_q(4)$, $\zeta + q(6), \ldots$, because the functions $1 + c_k \zeta_q(k)$ with suitable $c_k \in \mathbb{Q}$ are Eisenstein series for each even $k \geq 2$.

Theorem. For each $k \geq 0$, the function $\zeta_q(k+1)$ analytic on the domain $|q| < 1$ is transcendental over $\mathbb{C}(q)$. In essence, this result is an application of problems from [*G. Pólya* and *G. Szegő*, Aufgaben und Lehrsätze aus der Analysis, 2. Band Funktionstheorie. Nullstellen. Polynome. Determinante. Zahlentheorie, Springer-Verlag, Heidelberg-Berlin (1925; JFM 51.0173.01), Division 8].

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