Coclite, G. M.; Holden, H.; Karlsen, K. H.
Global weak solutions to a generalized hyperelastic-rod wave equation. (English)
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The Camassa-Holm equation is a nonlinear dispersive wave equation that takes the form

$$\frac{\partial u}{\partial t}-\frac{\partial^3 u}{\partial t\partial x^2}+2\kappa\frac{\partial u}{\partial x}+3u\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+u\frac{\partial^3 u}{\partial x^3},\quad t>0,\ x\in\mathbb{R}. \tag{1}$$

When $\kappa>0$, this equation models the propagation of unidirectional shallow water waves on a flat bottom, and $u(t,x)$ represents the fluid velocity at time t in the horizontal direction x. The Camassa-Holm equation possesses a bi-Hamiltonian structure (and thus an infinite number of conservation laws) and is completely integrable. Moreover, when $\kappa=0$ it has an infinite number of solitary wave solutions, called peakons due to the discontinuity of their first derivatives at the wave peak. The solitary waves with $\kappa>0$ are smooth, while they become peaked when $\kappa\to 0$. From a mathematical point of view the Camassa-Holm equation is well studied. Local well-posedness results have been proved. It is also known that there exist global solutions for a particular class of initial data and also solutions that blow up in finite time for a large class of initial data.

The authors of the paper are interested in the Cauchy problem for the nonlinear equation

$$\frac{\partial u}{\partial t}-\frac{\partial^3 u}{\partial t\partial x^2}+\frac{\partial}{\partial x}\left(\frac{g\left(u\right)}{2}\right)=\gamma\left(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+u\frac{\partial^3 u}{\partial x^3}\right),\quad t>0,\ x\in\mathbb{R}. \tag{2}$$

where the function $g:\mathbb{R}\to\mathbb{R}$ and the constant $\gamma\in\mathbb{R}$ are given. If $g(u)=2\kappa u+3u^2$ and $\gamma=1$, then (2) is the classical Camassa-Holm equation. With $g(u)=3u^2$, this equation describes finite length, small amplitude radial deformation waves in cylindrical compressible hyperelastic rods, and is often referred to as the hyperelastic-rod wave equation. The constant γ is given in terms of the material constants and the prestress of the rod. The authors coin equation (2) the generalized hyperelastic-rod wave equation. From a mathematical point of view the generalized hyperelastic-rod wave equation (2) is much less studied than (1). The existence of a global weak solution to (2) for any initial function u_0 belonging to $H^1(\mathbb{R})$ is established in the paper. Furthermore, the authors prove the existence of a strongly continuous semigroup, which in particular implies stability of the solution with respect to perturbations of data in the equation as well as perturbation in the initial data. The approach is based on a vanishing viscosity argument, showing stability of the solution when a regularizing term vanishes. This stability result is new even for the Camassa-Holm equation. Finally, the authors prove a "weak equals strong" uniqueness result.

Reviewer: Leonid B. Chubarov (Novosibirsk)

MSC:

35Q72	Other PDE from mechanics (MSC2000)
35D05	Existence of generalized solutions of PDE (MSC2000)
35G25	Initial value problems for nonlinear higher-order PDEs
74D10	Nonlinear constitutive equations for materials with memory
74H20	Existence of solutions of dynamical problems in solid mechanics
74K10	Rods (beams, columns, shafts, arches, rings, etc.)

Keywords:

Camassa-Holm equation; solitary wave; peakons; well-posedness; vanishing viscosity argument; weak equals strong uniqueness

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