Okounkov, A.; Pandharipande, R.
The equivariant Gromov-Witten theory of $\mathbb{P}^1$. (English) [Zbl 1105.14077]

The authors give an explicit description of the equivariant Gromov-Witten theory of $\mathbb{P}^1$. Namely, they prove that the equivariant Gromov-Witten potential of $\mathbb{P}^1$ satisfies the 2nd Toda equation. In particular, all Gromov-Witten invariants of positive degree are determined by those of degree zero. This result also completes the proof of the GW/H-correspondence introduced in their previous article [Ann. Math. (2) 163, 517–560 (2006; Zbl 1105.14076)].

The proof is done in several steps. The key point is to prove that the generating function of the equivariant GW-invariants of $\mathbb{P}^1$ equals a certain vacuum expectation value on the infinite wedge product (see Theorem 3). In order to prove Theorem 3, the authors proceed as follows:

First, using the equivariant localization formula of T. Graber and R. Pandharipande [Invent. Math. 135, 487–518 (1999; Zbl 0953.14035)], they express the generating function of the degree $d$ equivariant GW-invariants of $\mathbb{P}^1$ in terms of the Hodge integrals over the moduli space $\overline{M}_{g,n}$ (see Proposition 1).

Secondly, using the ‘ELSV-formula’ obtained by T. Ekedahl, S. Lando, M. Shapiro and A. Vainshtein [Invent. Math. 146, 297–327 (2001; Zbl 1073.14041)], and then dealing with complex analytical issues, the authors prove that the generating series for the $n$-point Hodge integrals equals the vacuum expectation of a certain operator on the infinite wedge space (see Theorem 2).

With the help of the formula given in Theorem 3, they establish that the Gromov-Witten potential satisfies the 2nd Toda equation (see Theorem 5 and 7).

The results of this article are further used by the same authors in [Invent. Math. 163, 47–108 (2006; Zbl 1140.14047)] to establish the Virasoro constraints for the relative Gromov-Witten theory of smooth target curves.

Reviewer: Mihai Halic (Zürich)

MSC:

14N35 Gromov-Witten invariants, quantum cohomology, Gopakumar-Vafa invariants, Donaldson-Thomas invariants (algebro-geometric aspects)
14N10 Enumerative problems (combinatorial problems) in algebraic geometry
14H70 Relationships between algebraic curves and integrable systems
37K10 Completely integrable infinite-dimensional Hamiltonian and Lagrangian systems, integration methods, integrability tests, integrable hierarchies (KdV, KP, Toda, etc.)

Keywords:
equivariant Gromov-Witten theory; Hurwitz theory; Toda hierarchy

Full Text: DOI arXiv