Zudilin, V. V.

On the irrationality of $\zeta_q(2)$. (English. Russian original) Zbl 1109.11316

Russ. Math. Surv. 56, No. 6, 1183-1185 (2001); translation from Usp. Mat. Nauk 56, No. 6, 147-148 (2001).

From the text: For complex q, $|q| < 1$, we define the quantity

$$\zeta_q(2) := \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} = \sum_{n=1}^{\infty} \sigma(n) q^n; \quad \lim_{\substack{q\to 1\\ |q|<1}} (1-q)^2 \zeta_q(2) = \frac{\pi^2}{6},$$

where $\sigma(n)$ is the sum of divisors of the positive integer n. The following is proved: Theorem 1. When $q = 1/p$, $p \in \mathbb{Z}\setminus\{0, \pm 1\}$, the number $\zeta_q(2)$ is irrational and its index of irrationality satisfies the inequality $\mu(\zeta_q(2)) \le 4.07869374\ldots$.

The q-arithmetic scheme and the q-hypergeometric construction of approximating linear forms also enable us to sharpen the known measures of irrationality for the quantities

$$\zeta_q(1) = \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}, \quad \ln_q(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^n}{1-q^n}, \quad |q| < 1,$$

which are the q-analogues of the (divergent) harmonic series and $\log 2$, respectively.

Theorem 2. For $q = 1/p$, $p \in \mathbb{Z} \setminus \{0, \pm 1\}$, the indices of irrationality of the numbers (3) satisfy the inequalities $\mu(\zeta_q(1)) \le 2.49846482\ldots$, $\mu(\ln_q(2)) \le 3.29727451\ldots$.

MSC:

11J82 Measures of irrationality and of transcendence
11M41 Other Dirichlet series and zeta functions

Full Text: DOI